POC II – Worksheet 4

Error Bounds in Circumference Estimates

Over two thousand three hundred years ago, Archimedes studied the circumference of the circle in his famous work, "Measurement of a Circle". This work of Archimedes is one of the first great analytical works in the history of our subject. The worksheet follows the spirit of Archimedes' approach and studies an approximation scheme for the circumference of the circle. The questions will give you the opportunity to explore how to use geometric reasoning to make estimates and determine the convergence, as well as the rate of convergence, of sequences in the context of at least partially answering an ancient question: How to define and measure the circumference of a circle.

Question 1.

Take $\mathscr{C}$ to be the unit circle and take $g_4(0)$ to be the square that is inscribed in $\mathscr{C}$ with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Sketch $\mathscr{C}$, $G_4(0)$, and $g_4(0)$ using Desmos.

Question 2.

To refine $g_4(0)$, project the midpoints of each edge of $g_4(0)$ onto $\mathscr{C}$. The refined polygon $g_4(1)$ is the inscribed polygon with vertex set equal to the vertex set of $g_4(0)$ together with the projected midpoints. Refine $g_4(1)$ in a similar way to obtain $g_4(2)$, and so on, so that $g_4(n)$ is an inscribed polygon with $4 \cdot 2^n$ edges. Sketch $g_4(1)$, $g_4(2)$, and $g_4(3)$ using Desmos. Hint: Calculate the position of as few points as possible, you will not need to make many calculations.

Question 3.

The convergence of the perimeter of $g_4(n)$ to the circumference of $\mathscr{C}$ seems to be quite fast. The figure below will help us to determine exactly how fast this convergence is. Figure 1 below displays edges that come from three stages of refinements of both an inscribed and a circumscribed polygon.

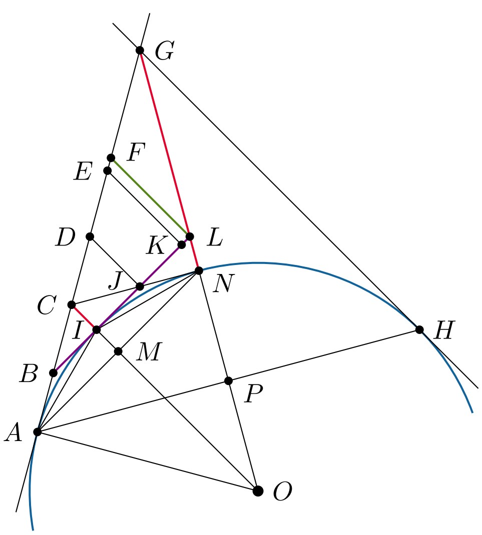

Figure 1

Take $G_4(0)$ to be the square that circumscribes $\mathscr{C}$, intersecting $\mathscr{C}$ at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. For each n, take $G_4(n)$ to be the regular circumscribed polygon that intersects $\mathscr{C}$ at the vertex set of $g_4(n)$. Roughly sketch by hand the circumscribed polygons $G_4(0)$, $G_4(1)$, $G_4(2)$, and $G_4(3)$.

Question 4.

In the Figure 1, take $\overline{AH}$ to be an edge of $g(n)$ and determine how $\overline{GH}$ and $\overline{AG}$ relate to $G_4(n)$. What line segments in the figure correspond, and how do they correspond, to edges or half edges of $G_4(n)$, $G_4(n+1)$, $G_4(n+2)$, $g_4(n)$, $g_4(n+1)$, and $g_4(n+2)$?

Question 5.

Take L_n to be an edge length of $G_4(n)$ and ℓ_n to be an edge length of $g_4(n)$. Take $h_4(n)$ to be the length of $\overline{NG}$. Calculate the height $h_4(0)$, the height of a vertex of the square $G_4(0)$ over $\mathscr{C}$.

Question 6.

Use Figure 1 to show that

$$2L_{n+1} < L_n, \quad \text{and} \quad 2\ell_{n+1} > \ell_n.$$

What do these inequalities imply about the perimeter of $G_4(n+1)$ compared to the perimeter of $G_4(n)$ and the perimeter of $g_4(n+1)$ compared with the perimeter of $g_4(n)$?

Question 7.

Find a relationship between ℓ_n and L_n and use this relationship together with the facts that you have discovered about the perimeters of the inscribed and circumscribed polygons to show that $(g_4(n))$ and $(G_4(n))$ are convergent.

Question 8.

Use Figure 1 to argue that

$$h_4(n+1) < \left(\frac{1}{3}\right) h_4(n)$$

and then, using the height of $h_4(n)$, estimate $h_4(n)$ and show that $h_4(n)$ tends to 0 as n tends to infinity.

Question 9.

Use Figure 1 to relate the sum $\ell_{n+1} + h_4(n)$ to $\frac{1}{2}L_n$. Use this relationship, together with the fact that

$$\frac{1}{2}L_n > L_{n+1} \quad \text{and} \quad L_{n+1} > \ell_{n+1}$$

to estimate $L_{n+1} - \ell_{n+1}$.

Question 10.

Estimate the difference in the perimeters of $G_4(n+1)$ and $g_4(n+1)$ and the rate at which the difference tends to 0. What does this tell you about the circumference of $\mathscr{C}$ if it is defined to be a length always greater than the perimeter of $g_4(n)$ and always less that the perimeter of $G_4(n)$?

POC II – Worksheet 8

Derivatives and the Graphical Representation of a Function

Sketching a function is an imprecise but important way of consolidating and communicating information about the function. This worksheet will walk you through the layered process of sketching a function given asymptotic and local information. Progressively introducing first order information and second order information refines a rougher asymptotic description of f. The function may be further specified by point-wise evaluation. Each layer of information specifies the function in a different way. Evaluation at a point, for example, exactly determines the function at a point, but does not necessarily give any information about the function at any other points, even if the points are nearby. By combining layers of information, we can better understand a function's behavior.

In the questions below, use the notation $f \underset{a}{\sim} g$ to mean that

$$f \in \underset{x\to a}{\mathrm{O}}(g(x)) \quad \text{and} \quad g \in \underset{x\to a}{\mathrm{O}}(f(x)),$$

and recall that f has a degree n zero at a point x_0 if $f \underset{x_0}{\sim} (x - x_0)^n$. Each function given below that is labeled by g is taken to be a different function in subsequent question. The function f is the same function in each of the questions and subsequent questions reveal more information about f.

Question 1.

Darboux's theorem states that if g is differentiable on an interval $[a, b]$, then g' has the intermediate value property on $[a, b]$. Why is this theorem not just an immediate consequence of the intermediate value theorem?

Question 2.

Darboux's theorem is important for determining the sign of the derivative of a function on an interval given information about the zeros of the derivative and the sign of the derivative at a point. Suppose that g is differentiable on $[-4, 7]$, that $g'(-1)$ and $g'(3)$ are both zero, and that g' has no other zeros in $[-4, 7]$. Given that $g'(-3)$ and $g'(1)$ are both negative and that $g'(7)$ is positive, what can you say about the sign of g' on subintervals of $[-4, 7]$? Find all maximal intervals on which g is increasing and all maximal intervals on which g is decreasing.

Question 3.

Suppose that g is continuous on $[-4, 7]$ and differentiable on $[-4, 2) \cup (2, 7]$. Suppose further that $g'(1)$ is negative, that $g'(3)$ is positive, and that g' is never equal to 0. Find all maximal intervals on which g is increasing and all maximal intervals on which g is decreasing.

Question 4.

Suppose that $g \underset{1}{\sim} (x - 1)^2$. If g is continuous, then what can g look like in a small interval near 1? Is it possible for g to be increasing immediately to the left and immediately to the right of 1, or decreasing immediately to the left and right of 1? Think carefully about what it means to be $\mathrm{O}((x - 1)^2)$ and keep in mind that g need not be a polynomial or a rational function.

Question 5.

Suppose that g is continuous on $[1, 5]$, twice differentiable on $[1, 3) \cup (3, 5]$, and that $g(3)$ is equal to 2. Suppose further that g'' is positive on $[1, 3) \cup (3, 5]$, that g' is positive on $[1, 3]$, and that g' is negative on $(3, 5]$. Sketch g. Is it possible for a function to have such a derivative and second derivative and be differentiable at 3?

Question 6.

The function g is defined and twice differentiable on $[1,3]$. Furthermore, g' is negative on $[1,3]$, g'' is positive on $[1,3]$, and

$$g(1) = 2 \quad \text{and} \quad g(3) = -4.$$

Is it possible for $g(2)$ to be equal to -1?

Question 7.

Suppose that the domain of f is $(-\infty,3)\cup(3,\infty)$ and that f is continuous at every point of $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$. Suppose further that

$$f \underset{-\infty}{\sim} x^3, \quad f \underset{\infty}{\sim} x, \quad f \underset{-4}{\sim} \frac{1}{x}, \quad f \underset{3}{\sim} \frac{1}{x^2}.$$

What might f look like? Be sure to describe all of the possibilities.

Question 8.

Suppose, in addition, that f has a degree two zero at 4, a degree 1 zero at 1, and a degree 3 zero at -2, but has no other zeros. What can f look like?

Question 9.

Suppose, in addition, that f is not differentiable at -4, 0, and 3. Suppose further that the zero set of f' is the set $\{-5,-2,4,6,8\}$ and that f' is positive on the set $\left\{-\frac{9}{2},-3,-1,\frac{7}{2},7\right\}$ and negative on the set $\{-10,1,5,11\}$. What can f look like?

Question 10.

Suppose now that f is twice differentiable on $(-\infty,-4)\cup(-4,0)\cup(0,3)\cup(3,\infty)$. Suppose further that the zero set of f'' is the set $\{-7,-6,-2,2,5,7,9\}$ and that f'' is positive on the set $\left\{-11,-5,-1,\frac{3}{2},6,12\right\}$ and negative on the set $\left\{-\frac{13}{2},-3,\frac{5}{2},4,8\right\}$. What does f look like?

Question 11.

Use the equalities

$$f(-7) = 5,\ f(-5) = 2,\ f(-4) = 1,\ f(0) = 2,\ f(2) = -1,\ f(6) = -7,\ f(8) = -1,\ \text{and}\ f(9) = -7,$$

to sketch f even more accurately.

Question 12.

Suppose that

$$f'(9) = -2.$$

Given that x is in $(9,\infty)$, describe all possible values for $\frac{f(x)}{x}$.

POC II – Worksheet 9

Telescoping Series and the Mean Value Theorem

A closed form for a sequence (s_n) of partial sums of an infinite series s is a way of writing the partial sums as a function of n, where the function is given by finitely many sums, products, and compositions of elementary functions. The geometric series is an important example of a series that has a closed form for its sequence of partial sums.

One difficulty in working with infinite series is that they seldom have such closed forms. An important way to determine the convergence or divergence of a series is to compare the series to one with a closed form, where questions about convergence and divergence are easier to answer. Telescoping series are the primary example of series that have closed forms. This worksheet will give you the opportunity to explore how the mean value theorem can facilitate the comparison of a given series with one that has a closed form.

Question 1.

For all questions below, suppose that r is in $(0, 1)$. Suppose for this question that f and g are given by

$$f(x) = x^r \quad \text{and} \quad g(x) = x^{-r}.$$

For each x in $[1, \infty)$, calculate $f'(x)$ and $g'(x)$. It may be helpful to experiment by initially taking r to be equal to, for example, $\frac{1}{3}$ for this question as well as those that follow.

Question 2.

Take p to be either be 1, $1 - r$, or $1 + r$. Recall that for all x in $(0, \infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}.$$

Take f to be the function that is defined for all x in $[1, \infty)$ by

$$f(x) = \frac{1}{x^p}.$$

Use the mean value theorem to determine all functions F with the property that

$$F'(x) = f(x).$$

Repeat this problem, but where f is given by

$$f(x) = \frac{1}{x(\ln(x))^p}.$$

It will help you to use the reverse chain rule for the second part of the problem.

Question 3.

Suppose that a and b are real numbers and that b is greater than a. Take F to be the function given by

$$F(x) = x^3.$$

The mean value theorem guarantees that there is a constant c in (a, b) so that

$$F'(c)(b - a) = F(b) - F(a).$$

Solve for c in terms of a and b. Repeat this problem but when a is 5, and b is 7, and F is the natural logarithm.

Question 4.

Take F to be the function given by

$$F(x) = |x|.$$

Although the function F is differentiable everywhere except at only one point, 0, then the conclusion of the mean value theorem will not be valid for F. Find an example that demonstrates this fact. Why does your example not contradict the mean value theorem?

Question 5.

Suppose that M is a natural number and that F is continuous on $[M, \infty)$, differentiable on (M, ∞), and that the derivative of F is f. Show that, for each natural number n that is greater than or equal to M, there is a c_n in $(n, n+1)$ such that

$$f(c_n) = F(n+1) - F(n).$$

Take N to be greater than M. Rewrite the sum

$$\sum_{n=M}^{N} f(c_n)$$

as a telescoping series.

Question 6.

Given the hypotheses of the previous question find a criterion for F so that infinite series

$$\sum_{n=M}^{\infty} f(c_n)$$

is (a) convergent, and (b) divergent.

Question 7.

Use the mean value theorem to compare the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+r}}$$

with a telescoping series and show that the infinite series is convergent.

Question 8.

Use the mean value theorem to compare the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-r}}$$

with a telescoping series and show that the infinite series is divergent.

Question 9.

Show that series that is given by

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

is convergent if p in $(1, \infty)$, and divergent if p in $(0, 1]$.