POC III – Worksheet 4

The Local Higher Order Approximation of a Function

The local higher order approximation of a function uses information about the higher derivatives of a function to approximate the function by a polynomial. It is a natural extension of the local linear approximation of a function to functions with more than one derivative. The more information about the derivatives of a function that are known, the better the local approximations can be. When the function is not only infinitely differentiable, but is also analytic on an interval and the Taylor series for the function is convergent on the interval, the derivatives of the function at a single point in the interval completely determine the function. This is a rather remarkable example of rigidity.

This worksheet will give you the opportunity to graphically explore the way that Taylor polynomial approximate a function and to computationally work with Taylor polynomials to approximate a function at points near a known value. It will also give you the opportunity to study the subtle interplay between rigidity and approximation.

Question 1.

Take f to be the function given by

$$f(x) = \frac{1}{4}\left(x^4 - 4x^3 - x^2 + 7x + 1\right).$$

Calculate the Taylor series for f that is centered at a. Take P_n to be first n terms of this Taylor series for f. How many terms of the Taylor series are required for the series to be equal to f? Use Desmos to sketch f and all of its Taylor polynomial approximations. Use a slider for a to see how these polynomial approximations change as you change the centering and be sure to sketch the point $(a, f(a))$ so that you can see where the polynomial approximations are centered.

Question 2.

Calculate the Taylor series for $\sin$ that is centered at a. Take P_n to be first n Taylor polynomial approximations centered at a for $\sin$. Use Desmos to sketch P_1, P_2, P_3, through P_{12}. Use a slider for a to see how these polynomial approximations change as you change the centering and be sure to sketch the point $(a, f(a))$ so that you can see where the polynomial approximations are centered. What does P_2 look like when a is equal to $\pm\frac{\pi}{2} + 2\pi k$ where k is an integer? Why is this the case?

Question 3.

Take f to be the function given by

$$f(x) = \frac{1}{1-x}.$$

Calculate the Taylor series for f that is centered at a. Take P_n to be first n Taylor polynomial approximations centered at a for f. Use Desmos to sketch P_1, P_2, P_3, through P_{12}. Use a slider for a to see how these polynomial approximations change as you change the centering and be sure to sketch the point $(a, f(a))$ so that you can see where the polynomial approximations are centered.

Question 4.

Suppose that P_n is a degree n Taylor polynomial for $\sin$ that is centered at a. Use the remainder term for P_n to determine how large n must be so that you can determine $\sin$ to within $.001$ of its actual value at the point $a + 4\pi$. Note that if you know $\sin(a)$, then you already know this value exactly since $\sin$ is 2π periodic.

Question 5.

Notice that even though the radius of convergence of $\sin$ is infinite, a high degree Taylor polynomial does not do a good job of approximating $\sin$ outside of a relatively small region. The periodicity of $\sin$ together with knowledge about certain values of $\sin$ and $\cos$ make it possible to come up with very good approximations of $\sin$. Approximate $\sin$ by using the symmetries of $\sin$ and by constructing a piecewise cubic polynomial P that is determined by degree three Taylor polynomials for $\sin$ centered at 0, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. What is the maximum possible difference between the approximation that you have constructed and $\sin$?

Question 6.

Your friend confidently claims that

$$.518 < \sin(31^\circ) < .52.$$

You have a calculator on hand that has π accurate to over 12 decimal places. Your calculator can perform basic algebraic operations, but it does not have buttons for the trigonometric functions. Using your calculator, show that your friend is incorrect.

Question 7.

In general, taking derivatives of a function results in rather complicated looking functions. So, in practice, it is quite difficult to evaluate the n^{th} remainder term for a Taylor polynomial approximation for a function in order to show that the function is analytic. However, sums and products of analytic functions are analytic. Given sufficient restrictions, so are quotients and composites of analytic functions. What would you guess these restrictions to be? Use these facts to show that the function f that is given by

$$f(x) = 1 + 5x + x^3 + 2x\sin\left(x^2\right),$$

is analytic on $\mathbb{R}$ and calculate its Taylor series centered at 0.

Question 8.

Suppose that f is infinitely differentiable in I and the Taylor series for f is convergent in I. The function f is not necessarily analytic. How might you detect that the function is not analytic by studying the remainder term that comes from the n^{th} Taylor polynomial for f?

Question 9.

By studying the remainder term for the n^{th} Taylor polynomial centered at 0 for f, where

$$f(x) = \ln(1 + x),$$

show that f is analytic on $(-1, 1)$. Show that the Taylor series for f given above does not converge when x is 2. Does this mean that f is not analytic in any open interval containing 2?

POC III – Worksheet 5

Constrained Motion

Motion of particles in physical systems is often constrained by the requirement that the particles in motion must maintain contact with a given body. Contact with the body is typically maintained by forces between the object and the body. These *contact forces* are equal and opposite to a constant (the particle's mass) times the component of the acceleration vector of the particle that is normal to the body at the point of contact.

The questions below will give you the opportunity to use the parameterization of a surface to explicitly parameterize the motion of a particle that is constrained to remain in contact with the surface. They will also give you the opportunity to study the acceleration of the particle and its relationship to the geometry of the surface.

Question 1.

Take S to be the sphere of radius R centered at the origin. Transform S into an ellipsoid E by applying a z-axis scaling to S to make the distance from $(0,0,0)$ to the highest point on E equal to s. Find an equation for E and use Geogebra to sketch E. Make s a slider and see what happens when you vary s.

Question 2.

A point p moves along E and has height h. Parameterize the motion of p, taking p to have a constant speed S and to move counterclockwise around the z-axis when viewed from above.

Question 3.

What is the velocity vector of p at time t, that is, at the point $p(t)$?

Question 4.

Use the equation for E to determine the level sets of E. Recall that the gradient of the function that determines the level sets of E is always perpendicular to the level set at a point on the level set. Use this fact to calculate a second vector that is tangent to E at $p(t)$ that is perpendicular to the velocity vector of p.

Question 5.

Find the equation for a small line segment $L(t)$ with midpoint $p(t)$ that is perpendicular to both the velocity vector of p at $p(t)$ and the normal vector to E at $p(t)$. Simulate the motion of p using Geogebra, together with an arrow with midpoint p to represent the velocity vector and the line segment $L(t)$. Your simulation should look like a small airplane flying along E.

Question 6.

What is the acceleration vector of p at $p(t)$ and what is the component of the acceleration that is normal to E at $p(t)$?

Question 7.

What is the component of the acceleration vector of p in the direction of the tangent vector of p at $p(t)$? Will this answer be true of any particle whose position is twice differentiable with respect to time and whose speed is constant?

Question 8.

Use Geogebra to animate the acceleration vector for p and experiment with different values of h.

Question 9.

How does changing the parameter s change the component of the acceleration vector that is perpendicular to E?

Question 10.

For what h will the acceleration vector be perpendicular to E? Can you think of anything that might be special about this particular path?

POC III – Worksheet 8

The Unit Circle and the Riemann Integral

The cosine, sine, and tangent functions of trigonometry are not only important in "real world" applications, they are also important in the theoretical development of our subject. A geometric development of these functions as functions of a radian angle measure that is sufficient for determining their local linear approximation requires the prior development of the notion of the length of an arc of the unit circle. Although there is an elementary approach for developing a notion of arc length that follows the approach of Archimedes and makes his approach rigorous from a modern perspective, the arguments are rather involved.

The Riemann integral is a powerful tool for constructing functions with specified properties. The questions in this worksheet will guide you through a development of the trigonometric functions that utilizes the Riemann integral and results in the determination of the length of circular arcs. The worksheet will give you the opportunity to better understand the interplay between calculus and geometry, and more deeply understand the importance of the Riemann integral.

Question 1.

Take m to be a real number in $[0, \infty)$ and take L_m to be the line with slope m that intersects the origin. Sketch the unit circle $\mathscr{C}$ and the line L_m overlaid on the x–y coordinate system. What are the coordinates of the point p_m in the first quadrant that is given by the intersection of L_m with $\mathscr{C}$? Set up a Riemann integral over the interval $[0, 1]$ to calculate the area of the sector of $\mathscr{C}$ that is defined by the arc from $(1, 0)$ to p_m.

Question 2.

The integral that you set up in the previous question will involve a piecewise defined function. Split this integral into two integral, one that gives the area of a triangle, and the other that is given by $I(m)$, where

$$I(m) = \int_{\frac{1}{\sqrt{1+m^2}}}^{1} \sqrt{1 - x^2}\, \mathrm{d}x.$$

Question 3.

Use the substitution

$$y = 1 - x^2$$

to rewrite $I(m)$ as a new integral $J(m)$.

Question 4.

The integral $J(m)$ above can be interpreted as an area that is calculated using horizontal rather than vertical rectangles in the Riemann sum approximations. The integrand is a function of the y–coordinate and so gives x–coordinates of points in terms of y–coordinates. Solve for the x–coordinates in terms of the y-coordinates, that is, compute the inverse of the integrand as a function of the x–coordinate.

Question 5.

Use the above calculations to justify that the area of the region given in the first question is $A(m)$, where

$$A(m) = \frac{1}{2} \int_0^m \frac{1}{1 + x^2}\, \mathrm{d}x.$$

Define the number π to be the constant with

$$\frac{\pi}{2} = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$$

and discuss (but do not carry out) a procedure for approximating this number to within an error of $\frac{1}{1000}$.

Question 6.

For each m in $[0, \infty)$, define

$$A(-m) = -A(m).$$

Show that A is differentiable on $(-\infty, \infty)$. Determine all inflections points of A.

Question 7.

Show that on $(-\infty, \infty)$, A has a well-defined inverse function, T.

(a) Determine the domain of T.

(b) Show that T is differentiable on its domain and use your sketch of A to sketch T.

Question 8.

Keep in mind for motivation that the goal is to define the trigonometric functions. Usually, the argument of the trigonometric functions is the length of an arc of a unit circle. More correctly, it is the length of an arc divided by the length of a radius, so the ratio has no units. The length of an arc of the unit circle is twice the area of the sector determined by the arc. Once again, more correctly, the length of an arc divided by the length of a radius is twice the area of the corresponding sector divided by the area of the square with side length given by the length of the radius, so that the ratio has no units. With this in mind, for each θ in $\left[0, \frac{\pi}{2}\right]$, take

$$\tan(\theta) = T\left(\tfrac{\theta}{2}\right).$$

(a) Calculate $T'(\theta)$.

(b) Use the chain rule to calculate $\tan'(\theta)$ in terms of $T\left(\frac{\theta}{2}\right)$ and then rewrite $\tan'(\theta)$ in terms of $\tan(\theta)$.

(c) Define $\cos$ and $\sin$ by

$$\cos(\theta) = \frac{1}{\sqrt{1+(\tan(\theta))^2}} \quad \text{and} \quad \sin(\theta) = \frac{\tan(\theta)}{\sqrt{1+(\tan(\theta))^2}}.$$

Use (b) and the chain rule to calculate $\cos'(\theta)$ and $\sin'(\theta)$.

Question 9.

For each θ in $\left[0, \frac{\pi}{2}\right]$, the point $(\cos(\theta), \sin(\theta))$ is a the point on the unit circle in the first quadrant so that the sector of the circle given by the arc from $(1,0)$ to $(\cos(\theta), \sin(\theta))$ has area equal to $\frac{\theta}{2}$. Take γ to be the path given by

$$\gamma(\theta) = (\cos(\theta), \sin(\theta))$$

that describes the motion of a particle that is at $(1,0)$ when θ is 0 and that moves counterclockwise along $\mathscr{C}$. What is the is the speed of the particle at any given time? What is the length of the arc from $(1,0)$ to $(\cos(\theta), \sin(\theta))$?

Question 10.

How will you define $\cos$, $\sin$, and $\tan$ for θ outside $\left[0, \frac{\pi}{2}\right]$? What have you accomplished in the course of answering the above questions and why is what you have done so important?