Worksheet 1

Visualizing Intervals and Containment

Physical intuition about space and geometry gives us a meaningful way to interact with abstract mathematical notions and make them relatable. This worksheet presents an opportunity for you to develop your natural ability to visualize abstract quantities and use visualization to determine containment and to solve inequalities. In exploring the questions below, draw pictures to help find physical meaning in the mathematical symbols and expressions.

Question 1.

Sketch on a number line the sets $(-\infty, 5]$, $[-4, \infty)$, $(2, \infty)$, and $(-\infty, 7)$.

Question 2.

Use the sketches above to write in interval notation and sketch on a number line the sets $[-4, \infty) \cap (-\infty, 5]$ and $(2, \infty) \cap (-\infty, 7)$.

Question 3.

For any two sets A and B, take $A \setminus B$ (reads: A without B) to be the set of all members of A that are not members of B. Use the sketches above to determine, write in interval notation, and sketch the sets S_1 and S_2 where

$$S_1 = \big([-4, \infty) \cap (-\infty, 5]\big) \cup \big((2, \infty) \cap (-\infty, 7)\big) \quad \text{and} \quad S_2 = \big([-4, \infty) \cap (-\infty, 5]\big) \cap \big((2, \infty) \cap (-\infty, 7)\big).$$

Sketch the set $S_1 \setminus S_2$ and express this set using interval notation.

Question 4.

Using the notation above, sketch the set S that is given by

$$S = \big(S_1 \setminus S_2\big) \cap (-1, 6].$$

Describe this set using interval notation. Note that decomposition of S into various pieces has allowed you to make sense of a set that was initially described in a very complicated way, namely

$$S = \Bigg(\Big(\big([-4, \infty) \cap (-\infty, 5]\big) \cup \big((2, \infty) \cap (-\infty, 7)\big)\Big) \setminus \Big(\big([-4, \infty) \cap (-\infty, 5]\big) \cap \big((2, \infty) \cap (-\infty, 7)\big)\Big)\Bigg) \cap (-1, 6].$$

Commentary: Mathematics can include some very complicated multi-layered definitions and questions. It can be bewildering to see a cluttered mass of symbols on a page! The first step in even reading such a complicated expression is to decompose the expression into manageable pieces and to develop intuition about the meaning of the expression.

Question 5.

Suppose that a is in $(0, 1)$. What is the relationship between $[0, a]$ and $[0, a^2]$?

Question 6.

Suppose that a is in $(1, \infty)$. What is the relationship between $[0, a]$ and $[0, a^2]$?

Question 7.

Suppose that b is a real number and that E is a positive real number. Find all possible values for E so that for all a in $(b-2E, b+3E)$, if x is in the interval $(a-E, a+E)$, then x is in $(b-2, b+2)$.

Question 8.

Suppose that b is a real number and that E is a positive real number. Find the least possible real number e so that for any a in $(b-E, b+E)$, if x is in $(-\infty, a-e) \cup (a+e, \infty)$, then x is not in $(b-E, b+E)$.

Question 9.

Take A, B, C, D, and E to be sets and suppose that

$$A \cap B \subseteq E \quad \text{and} \quad C \cap D \subseteq E.$$

Your friend confidently, and incorrectly, asserts that the following statement must be true:

$$(A \cup C) \cap (B \cup D) \subseteq E.$$

Find the simplest possible counterexample to this statement and then find a counterexample where the above sets are intervals.

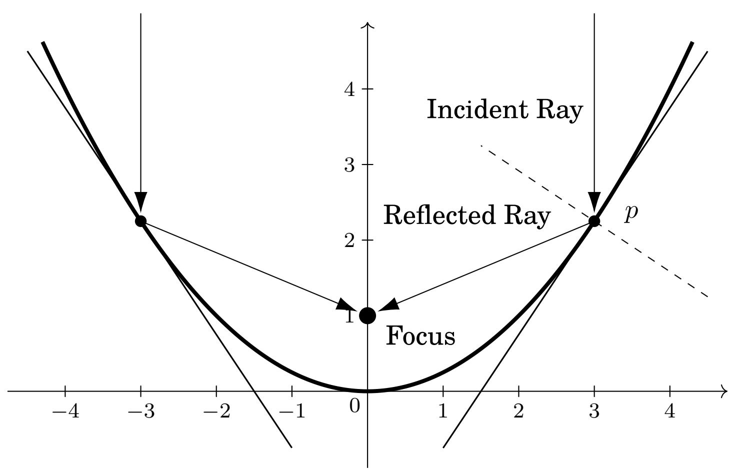
Worksheet 10

Telescopes and Tangency

In the year 1609, Galileo first pointed his little telescope upwards to gaze into the night sky. He studied the surface of the moon and found that it has a terrain marked by mountains and valleys. When he looked towards Jupiter, he found moons that orbit the planet and carefully recorded their motion. He saw surfaces features that changed periodically, showing that the planet itself rotated on an axis. He observed in the Jovian system a tiny model of our own solar system. At once, the heavens themselves became dynamic, and the reach of human understanding exploded far beyond the confines of our planet. Our world that had once seemed immeasurably vast, became microscopic.

Newton's invention of the reflecting telescope paved the way for the behemoth telescopes of modern times that have given us even more insights and inspired even further intellectual growth. This worksheet will study the reflection properties of parabolas and will help us to understand, at least at a basic level, how reflecting telescopes work. The questions will give you the opportunity to study an important application of mathematics in modeling a physical system. They will also give you the opportunity to see how the study of a concrete physical system can further our understanding of abstract mathematical objects.

We need one fact about reflections in two dimensions that we can regard as experimentally determined: When light falls onto a reflective curve, hitting a point p, it reflects in such a way that the incident ray and the reflected ray are reflections across the line intersecting p and perpendicular to the line tangent to the curve at p. The picture below illustrates this principle.



Question 1.

Take A to be a positive real number and take f to be the parabola given by

$$f(x) = Ax^2.$$

For the next few questions, take A to be equal to 3. Sketch the function f and draw at least three incoming (incident) light rays that are parallel to the axis of symmetry of f. Where should the incident light ray that moves downwards along the y-axis reflect?

Question 2.

Find an equation for the L that is tangent to f at $(2, 12)$ and find an equation for the line $L_\perp$ that is perpendicular to L and that intersects $(2, 12)$.

Question 3.

Find an equation for the path of motion of the light ray that starts at a point above $(2, 12)$, moves vertically down to $(2, 12)$, and reflects off of the parabola.

Question 4.

Take a to be a real number, but do not specify what it is. Find an equation for the path of motion of the light ray that starts at a point above $(a, f(a))$, moves vertically down to $(a, f(a))$, and reflects off of the parabola. Note that, regardless of a, all paths of motion will intersect the y-axis at a single point, the *focus* of f. Determine the focus of f.

Question 5.

For this and the following questions, no longer require A to be 3 and leave it instead as an undetermined positive coefficient. Redo the previous problems.

Question 6.

Make a slider A and use Desmos to sketch the function f. It will be better to have A take on small positive values, so restrict the range of the slider for A with this in mind.

Question 7.

Make a sider a and sketch on Desmos the path of motion of a light ray that starts at a point above $(a, f(a))$, moves vertically down to $(a, f(a))$, reflects off of the parabola, and then moves onward to the focus of the parabola. Repeat this with a new slider labeled b and have both light rays start at the same height h. It would be useful to construct a slider for h so that the height may be changed.

Question 8.

Parameterize both paths of motion given by the previous question in such a way that they describe particles that move at unit speed along the paths and begin at the same height h above the parabola but with different first coordinates. Simulate the motion of these particles in Desmos. The particles start at different points and may hit the parabola at different times, but when do they meet at the focus? What does this tell you about how far each has gone?

Question 9.

For each of the particles you have simulated, construct another "paired" particle that starts at the same place and with the same constant velocity, but that passes through the parabola and continues to move downward. Determine the horizontal line that the non reflecting particles will strike exactly when their "paired" particle strikes the focus. This horizontal line is the *directrix* of the parabola. Sketch the directrix and simulate the particles' motion. With the simulation running, change the slider for a, b, and A. You will notice that the focus and directrix also change.

Question 10.

Conjecture and prove a theorem that relates the parabola given by f, the focus of f, and the directrix of f.

Commentary: In the questions above, we modeled light falling into a reflecting telescope as incoming parallel light rays that were parallel to the axis of symmetry of the parabola. This is essentially correct if we are pointing the telescope at a star and the star lies along the axis of symmetry of the parabolic mirror: The star is very far away and the resolution of the telescope is too low to see that the star is different from a point source of light. If multiple stars are in view, they cannot all simultaneously lie along the axis of symmetry. In this case, the model is more complicated. Telescope designers (and users) must contend with these complications.