The Principles of Calculus III

Oct 14, 2021

VII. Local Higher Order Approximation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a series of functions.
Recall the definition of a power series.
Recall the definition of radius of convergence.
Recognize some common convergent and divergent series of functions and power series.
Recognize that statement of a simplified version of the Weierstrass Approximation Theorem.

Application and Analysis (Analysis)

Use the comparison test, limit comparison test, M test, and simplified versions of the root and ratio tests.
Use simplified versions of the root and ratio tests to determine the radius of convergence of a power series.
Use the alternating series test to determine the convergence of an alternating series and an alternating power series.

Evaluate and Create (Synthesis)

Estimate error terms for convergent alternating series.
Determine convergence using comparison tests.
Determine radius of convergence of a power series.

Summary.

This section presents the notion of convergence of a series as the convergence of the sequence of partial sums. This section is critical for the following section on Taylor series.

This section also presents the polynomial approximation of continuous function, the Weierstrass approximation theorem, as an example of a global approximation rather than a local approximation. The study of the Weierstrass approximation theorem ties together students' previous work (in exercises and worksheets) on local linear approximation of continuous functions, expressing continuous piecewise linear functions as a sum of absolute value functions, and the Babylonian approximation of the square root function.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Recall the definition of a Taylor polynomial and of Taylor's theorem.
- Recall the meaning of higher order derivatives.
- Recall the definition of the Taylor series for a function and the radius of con.
- Recall that two functions that are analytic on an open interval if they are equal on a convergent sequence in the open interval.

Application and Analysis (Analysis)

- Calculate the Taylor series for a polynomial.
- Calculate the Taylor series for a functions.
- Use the second derivative test.
- Determine the shape of a function based on concavity and extremal points (revisit the topic and emphasize non-algebraic functions as well as higher order information and osculating circles).
- Determine the radius of convergence of a Taylor series.

Evaluate and Create (Synthesis)

- Determine the error from remainder terms in Taylor's theorem.
- Derive the main rigidity theorem for real analytic functions, that analytic functions that are zero on a convergent sequence are identically zero.

Summary.

We begin our study of Taylor polynomials by studying the Taylor polynomial associated to a polynomial function, where there are only finitely many non-zero terms. We then extend this study to the study of Taylor polynomials and Taylor series for more general functions. This section is critical for understanding the higher order approximation of functions, for more accurately sketching functions, and for the approximations of integrals that will appear later.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall sufficient conditions for term-by-term differentiation of a power series.
Recognize a linear higher order differential operator.
Recognize how a linear higher order differential operator acts on a power series.

Application and Analysis (Analysis)

Determine the series solutions to some simple higher order differential equations.

Evaluate and Create (Synthesis)

Justify the existence of antiderivatives in certain restricted settings using power series.
Evaluate the limitations of using power series to define antiderivatives.

Summary.

It is important for students to go beyond learning about the formal properties of power series and to understand some of their important applications. This chapter studies some of their applications to the study of differential equations. This section also introduces students to the important idea that not all function that arise in applications have a closed form and that it is absolutely vital to be able to approximate functions, provide efficient approximation schemes, understand rates of convergence, and, in modern applications, use computing resources to work with functions that may be determined in complicated ways.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a higher order derivative of a curve.
Recall the meaning of force and acceleration for particles moving in the plane and in three dimensional Euclidean space.
Recall the definition of the dot product.
Calculate the component of force, velocity, and acceleration in a given direction.

Application and Analysis (Analysis)

Determine paths of motion of particles restricted to planes given by graphs of functions, including rotations around a point in the plane.
Determine paths of motion of particles restricted to surfaces given by graphs of functions.
Determine paths of motion of particles restricted to surfaces given by general surfaces.
Calculate the equation of the tangent plane to a surface at a given point.
Determine the tangent and normal bundles of a surface.

Evaluate and Create (Synthesis)

Calculate the component of acceleration of a particle moving on a surface that is tangent to the surface.
Calculate the normal acceleration of a particle whose motion is restricted to a surface.

Summary.

This is a culminating section on the geometry of particle motion. It reinforces the ideas needed for the final section of the course that deals with a simplified version of Green’s theorem and looks ahead to the multivariable calculus courses.

VIII. Integration

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the Fundamental Theorems of Calculus I and II.
Recall the Integral Mean Value Theorem
Recognize when integrals are improper integrals.

Application and Analysis (Analysis)

Calculate the average value of a function on an interval.
Calculate the area between two curves in the plane.
Calculate the derivatives of integrals with variable bounds.
Use upper and lower Riemann sums, the midpoint rule, the trapezoid rule, and Simpson's rule to approximate the value of an integral and the area between two curves in the plane.
Evaluate improper integrals.
Associate the integral test for convergence of a series to the use of the mean value theorem in determining the convergence of a series.

Evaluate and Create (Synthesis)

Determine estimates for the error terms that arise from approximation of integrals.
Determine estimates for the error terms arising from improper integrals.

Summary.

We use the notion of the Riemann integral, already established in the first section of the course, to make rigorous the notion of the area bounded by a Jordan curve given by the graph of a Riemann integrable function. We prove the FTC I and II and use the integral to prove the existence of an antiderivative for any continuous function. Practice with differentiating integrals with bounds that are functions of a single real variable reinforce students understanding of both the chain rule and the fundamental theorem of calculus.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Define the arc length of a curve.
Recognize the difference in difficulty in determining the area bounded by an ellipse and determining the length of an arc of a parameterized ellipse.
Identify some functions whose antiderivatives are not elementary.
Recognize a library of functions that are defined as definite integrals over a variable domain.
Recognize an expanded library of functions.
Recall simple mollifiers, smoothing kernels, etc.

Application and Analysis (Analysis)

Key Point: Reinforce the FTC.
Calculate the arc length of a parameterized curve in the plane and in three dimensional euclidean space.
Use the properties of an integrand to determine the properties of a function that is defined as a definite integral over a variable domain.

Evaluate and Create (Synthesis)

Formulate approximation procedures to estimate the value of a function defined by an integral (including elementary functions defined in this way).
Create tables of approximate values for functions that are determined by integrals using computing technology.
Generate/Justify interpolation schemes to approximate the values at intermediate points.
(In textbook for reference, but for use in a class as an enrichment activity or if time permits) Establish the Leibniz rule and use it to derive values of certain definite and improper integrals: Justify that $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Summary.

There is an extreme an unfounded focus in current calculus courses on manipulating elementary functions and explicitly determining the antiderivatives of functions that have elementary antiderivatives. Risch's (semi)algorithm essentially solves the problem and so practicing these techniques are essentially a vestigial topic in our calculus courses. However, integration techniques are still important for manipulating functions that are not elementary, determining their values and properties, and so on.

They key pedagogical reason for this section is to reinforce student understanding of the FTC and see how it is different from the application of the mean value theorem in determining antiderivatives. Students will recognize (but do not prove or discuss the details in class) that there is a theorem that establishes that certain functions do not have elementary antiderivatives (Liouville's Theorem for Differential Algebras). Students will see that the arc length of an arc of an ellipse is a first example, and a compelling example, of a function that is not elementary. Students learn in this section that one of the main values of the Riemann integral is for defining new functions that solve certain differential equations and have certain properties. Every following section in this course will use such functions. For example, students can later use the various integration methods (substitution, IBP) to manipulate such functions (Gaussian integrals/the erf function, as a major example in probability; the sinc function; elliptic integrals of the second kind, as the motivating question). They will eventually see how the inverse trigonometric functions and logarithms may be defined by integrals, and then be used to determine the trigonometric functions and exponentials is a completely rigorous way. This will expand students' understanding of functions, vastly increase students' reach in exploring examples, and give students the opportunity to learn about the importance of computing technologies.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify when integration by parts is useful for calculating antiderivatives.
Recognize bijections between domains of integration.
Identify when substitution is useful for calculating antiderivatives.

Application and Analysis (Analysis)

Calculate basic examples of definite and indefinite integrals that use integration by parts.
Calculate basic examples of definite and indefinite integrals that use integration by parts.
Calculate basic examples of definite and indefinite integrals that use substitution.

Evaluate and Create (Synthesis)

Determine whether or not a real valued function on a subset of the line is a valid change of variables.
Justify the use of substitution in calculating definite and indefinite integrals.

Summary.

This section develops the main calculation tools in single variable integral calculus. We use integration by parts to prove Taylor's theorem with an integral remainder term. We carefully study the substitution theorem for Riemann integrals, correcting the usual incorrect presentation that views the integral as a directed integral and using instead the presentation that is in line with the theorem in the multivariable setting. We pay particular attention to identifying valid changes in variables. This approach is in line with the transformational approach of the precalculus course.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify polynomials that are irreducible in the real numbers.
Recall how to write a quadratic polynomial in scaled and shifted form.
Recall the change of variables needed to write the reciprocal of and quadratic polynomial in the form $\frac{1}{x^2+1}$.

Application and Analysis (Analysis)

Calculate the antiderivative of even powers of the cosine function.
Calculate the partial fraction decomposition of any rational fraction with a denominator in factored form.

Evaluate and Create (Synthesis)

Calculate indefinite and definite integrals of any rational function with a denominator in factored form.
Calculate improper integrals involving rational functions.

Summary.

Antiderivatives of elementary functions are seldom elementary. It is important that in teaching students integration techniques we make students aware of this fact. It is also important that they understand that calculating antiderivatives that are elementary functions is, for our intents and purposes, a solved problem. They should use and be encouraged to use symbolic solvers to do integration problems. They should also understand that large classes of problems can be easily solved by hand. Integrals of rational functions are among this class of problems. We teach the techniques with an emphasis on reinforcing students ability to apply the principle of decomposition, that is, to reduce a single complicated problem into several problems of a simpler type. Students should be aware that this is why we are teaching the topic at hand, not that it is independently important. Indeed, reduction of a problem to a partial fraction decomposition is a critical aspect of Risch's (semi)algorithm that symbolic solvers employ.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definitions of the hyperbolic trigonometric functions and their relationship to the circular trigonometric functions.
Recognize integrals that can be solved by trigonometric and hyperbolic trigonometric substitution.
Recall the birational equivalence between the unit circle and the y-axis.
Recall the birational equivalence between the right half of the hyperbola given by $x^2 - y^2 = 1$ and the y-axis between -1 and 1.

Application and Analysis (Analysis)

Calculate basic examples of definite and indefinite integrals involving trigonometric integrals.
Calculate basic examples of definite and indefinite integrals involving hyperbolic trigonometric functions.

Evaluate and Create (Synthesis)

Justify the use of the Weierstrass substitution and the use of its hyperbolic analog.
Formulate more advanced trigonometric and hyperbolic trigonometric substitutions, where more careful attention needs to be paid to the domain and ranges of the functions, to calculate definite and indefinite integrals.

Summary.

This section explores a general framework for trigonometric and hyperbolic trigonometric substitution from a transformational viewpoint. Often presented as a "sneaky" trick, the Weierstrass substitution is well motivated and comes from the birational equivalence of the circle and the y-axis. It identifies all rational points on the unit circle except $(-1, 0)$ with the rational points on the y-axis and gives a simple way of determining all Pythagorean triples. A similar transformation exists for the hyperbola given by the locus of points satisfying $x^2-y^2 = 1$. These transformations are very useful for evaluating complicated integrals and more naturally motivates the derivations of the anti-derivatives of the reciprocal trigonometric

functions. Studying these problems and the rational points on these curves opens the door for students to study rational points on elliptic curves, a topic that is not only important in its own right as a part of algebraic geometry, but also one that has become very important in computer science and cryptography.

The key point is that students will begin to understand that progress in mathematics is nonlinear and that exploration of ideas for their own sake in some areas can lead to startling applications in other areas. It also helps students to understand that one should seek to generalize ideas in order to solve broader classes of problems. Many isolated tricks can solve specific integration problems more easily and more efficiently, but we are generally searching for frameworks that solve all problems of a given type.

This section reflects our overall theme of avoiding highly specialized tools that give only minor advantages and have a narrow range of applications in favor of approaches that are general, powerful, and interesting in their own right.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the difference between the differential equations that govern the motion of ballistics with and without air resistance.

Match sketched regions of the plane to planar regions given by the intersection of feasible sets.

Identify symmetries of surfaces of revolution and the volumes they bound.

Recall the equation of the infinitesimal surface area element for a surface of revolution.

Recall the equation of the infinitesimal volume element for a surface of revolution.

Recall the definition of work done by a force field.

Application and Analysis (Analysis)

Calculate areas bounded by polar curves using symmetry.

Calculate the area bounded by general planar curves using a simplified form o f Green's theorem.

Calculate the work done by a force field in moving a particle from one point in the plane or three dimensional Euclidean space to another.

Evaluate and Create (Synthesis)

Justify using Grönwall's inequality the uniqueness of the solution of the differential equation for describing air resistance.

Simulate the differences in ballistic motion with and without air resistance and numerically approximate the differences.

Calculate the surface area of a surface of revolution.

Calculate the volume bounded by a surface of revolution.

Justify a simplified version of Green's theorem for certain vector fields.

Determine the area of a region determined by a Jordan curve with a continuously differentiable parameterization.

Summary.

Develop mathematical models for studying ballistics with wind resistance. We solve the problem of ballistics with air resistance as a capstone topic.

Study the areas bounded by polar curves by identifying "infinitesimal" area elements that are sectors of circles rather than rectangles. The topics of this section involving surfaces of revolution are rather standard, however, we pay particular attention to the identification of symmetries of surfaces of revolution to identify appropriate infinitesimal surface area and volume elements.

We conclude the course by calculating the area bounded by a continuously differentiable Jordan curve. This is also a capstone topic. The main point is that for vector fields $\frac{1}{2}\langle -y, x\rangle$, $\langle -y, 0\rangle$, or $\langle 0, x\rangle$ if Δ is a triangle and $\gamma\colon [0,1] \to \Delta$ is a counterclockwise oriented parameterization that is continuously differentiable at all but finitely many points, then the area of Δ is equal to $\int_0^1 F(\gamma) \cdot \gamma'(t)\,\mathrm{d}t$. The proof of Green's theorem for such vector fields requires only the tools that we have developed and solves an important and ancient problem.

The Principles of Calculus III

Lecture 1

Lecture 2– 3

Lecture 4

Lecture 5– 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Lecture 12– 13

Lecture 14

Lecture 15– 16

Lecture 17

Lecture 18

Lecture 19

Lecture 20

Lecture 21

Lecture 22

Lecture 23

Lecture 24

Lecture 25

Lecture 26

VIII.6.2. Application of Symmetry Principle: Area and Volume Integrals

Lecture 27

VIII.6.3. Application of Symmetry Principle: Surface Area

Lecture 28

VIII.6.4. Work Integrals