Principles of Calculus II

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The Principles of Calculus II

V. Finite Approximation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of area for a rectangle.
Recall the definition of a rectangular and triangular boundary.
Recall the definition of an oriented polygonal curve.

Application and Analysis (Analysis)

Calculate the area of a triangle using the shoelace formula.
Calculate the area of a triangle using Heron’s formula.
Determine the orientation of a rectangular and triangular curve.

Evaluate and Create (Synthesis)

Calculate the circumcircle of a triangle.
Identify whether a point is inside or outside quadrilateral.

Summary.

Students develop a notion of the area bounded by a rectangle and a triangle. These are the “simple” shapes that will eventually give rise to a notion of area in the plane bounded by certain curves.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall the definition of a polygonal Jordan curve.
Recall the statement of the Jordan Curve Theorem for polygonal curves.
Recall the meaning of the area of a triangle and the surveyor's (or shoelace) formula.
Identify a triangulation of a polygonal Jordan curve.

<u>Application and Analysis</u> (Analysis)

Demonstrate the existence of triangulations of a polygonal Jordan Curve.
Calculate the area of a simple closed polygonal curve using the surveyor's (or shoelace) formula.

<u>Evaluate and Create</u> (Synthesis)

Derive the additivity of area defined for polygonal Jordan curves.
Determine that this notion of area coincides with our physical intuition.
Defend the non-triviality of the area problem for general Jordan curves.

Summary.

Students develop a notion of the area bounded by a Jordan Curve. Although difficult to state in generality because of the need to define the notion of a homeomorphism, which is beyond what the students can understand at this point, we can without difficulty state all relevant results for polygonal curves. The general problem about calculating the area bounded by a Jordan curve can be stated roughly, alerting the students to the fact that a more careful statement is required for further study. Since we will only work with Jordan curves that are given by either graphs of functions together with three bounding line segments or, at the end of the course, for curves that are continuously differentiable except for finitely many points, our approach is sufficient. The key point is that students are already in the first lecture thinking about additive quantities, subdivision, superposition, and approximation of difficult to compute quantities by those that are more tractable, namely the area bounded by a Jordan curve as being approximated by a polygonal Jordan curve.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a sequence.
Recall the limit laws for sequences.
Recognize the limits for some simple sequences.
Explain non-rigorously why certain simple limits (ex. $\left(1+\frac{1}{n}\right) \to 1$) should be what they are and why certain limits (ex. $\left(1+\frac{1}{n}\right)^n \not\to 1$) are more difficult to compute.

Application and Analysis (Analysis)

Calculate simple limits using limit laws (including the squeeze theorem).
Calculate limits of recursively defined sequences.
Use error terms to compute more complicated limits.
Demonstrate when sequences have infinite limits.

Evaluate and Create (Synthesis)

Formulate statements equivalent to the Archimedean property of $\mathbb{R}$.
Formulate statements equivalent to the finite intersection property of $\mathbb{R}$.
Justify the existence of limits using basic facts about the real numbers.
Create sequences with specified limits.
Argue that certain sequences do not have limits.

Summary.

The notion of a sequential limit will be discussed in a rigorous way. All ensuing discussion of limits will be rigorously formulated in terms of sequential limits. Sequential limits are more concrete and easier to conceptualize. While this approach is very efficient, the efficiency comes at a cost. Namely, formulating the notion of a continuous limit requires a universal quantification over a collection of sequences. This is hard for students. However, limit laws become automatic in the continuous case from the sequential limit laws and showing the failure of the existence of a continuous limit is naturally framed as finding a specific sequence with certain properties.

• To Outline
• To Lectures

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Define a fraction of a circle.
Use the nested intersection property to define irrational "fractions".
Define approximation procedures for determining the area bounded by a circle and its circumference.

Application and Analysis (Analysis)

Calculate limits for the trigonometric functions with respect to different angle measures.

Evaluate and Create (Synthesis)

Determine upper and lower bounds arising from approximations of a circle.

Summary.

We review the basic ideas of Archimedes' famous work and show how the area bounded by a circle is defined and how the length of an arc of a circle is defined.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Identify the limits associated to some simple functions.
- Recall that limiting values of a function do not depend on the function value at the limit point.
- Recall the definition of a continuous limit in terms of sequential limits.
- Recall the definition of a one sided limit in terms of sequential limits and restriction of domain.
- Recall the definition of an infinite limit and a limit at infinity in terms of sequential limits.
- Recognize that the limit laws of continuous limits come from the limit laws for sequences.

Application and Analysis (Analysis)

- Use the error estimates to calculate limits.
- Use limit laws to calculate limits.

Evaluate and Create (Synthesis)

- Determine complicated limits.
- Use sequences to find counterexamples to the existence of a limit or to the validity of a proposed limit.
- Justify certain limits without using the $\delta - \varepsilon$ formalism and instead using the Landau notation and analysis of error terms.

Summary.

In the process of solving exercises in this section, students will already make the necessary calculations for determining the continuity of elementary functions and determining the derivatives of most of the basic elementary functions. This disentangles some of the technical difficulties from the later conceptual difficulties that students will face. For example, in the next section, they will need to show that sin, $\sqrt{\cdot}$, and other elementary function are continuous at every point x in their domains. In the previous section, they will have already estimated $\sin(x+h) - \sin(x)$, $\sqrt{x+h} - \sqrt{x}$, and other such differences. In this section, they will use their estimates from previous sections to calculate limits associated to the differences. In the next section, they will use limits to make and prove statements about continuity.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Recall the definition of continuity.
- Recall the intermediate value theorem and the fact that continuous functions attain their maximum and minimum values on closed and bounded intervals.
- Recognize a basic library of continuous functions.
- Identify the basic algebraic properties of continuous functions as properties that arise from the limit laws.

Application and Analysis (Analysis)

- Determine the asymptotic behavior of rational functions.
- Determine whether or not given functions (including piecewise defined functions) are continuous at specified points or on specified intervals.
- Demonstrate that certain functions are not continuous at specified points or on specified intervals.

Evaluate and Create (Synthesis)

- Generate functions with specified continuity properties and specified asymptotic properties.
- Create approximation schemes using the bisection method to find the solutions to equations.
- Justify the existence of solutions to equations involving continuous functions.

Summary.

Students learn to view continuity as a property that determines approximability. They learn to use limits to determine whether or not a function output is approximable at some point by nearby input values. They also learn to view this property graphically and use the properties of continuous functions to estimate solutions to equations involving continuous functions. Understanding the properties of continuous functions will be critical henceforth because we will generally restrict our study of functions to those that are continuous on open intervals.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize proper usage of the Landau symbols.
Describe error terms coming from differences of the form $f(x+h) - f(x)$.
Recall that differences of the form $f(x+h) - f(x)$ are differences coming from perturbation of input values.

Application and Analysis (Analysis)

Calculate upper and lower bounds arising from absolute values of functions.
Analyze error terms coming from sums and products of functions.

Evaluate and Create (Synthesis)

Estimate error terms coming from perturbing quotients and composites of functions.
Illustrate graphically the errors due to perturbing inputs of a function.

Summary.

The only limits that we directly study will be limits that tend to zero and infinite limits. This section studies limits that tend to zero and so provides a tool for studying continuity and for studying differentiability. We purposely avoid the $\delta - \varepsilon$ formalism, but make sure that all technical tools are in place and conceptual tools are established for students to later make use of this formalism. The reason for this is that understanding the formalism is really a two step process. First, students must be able to perform the algebraic manipulations. Second, they must understand the more abstract logical basis for the formalism. The second is difficult for students who lack the basic technical skills. Focusing on the technical skills is sufficient for applications to calculus, whereas the formalism is required for rigorous proofs, which come in later classes.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of the difference quotient.
Recognize the relationship between the difference quotient and the average rate of change of a function on an interval.
Recognize the form of the difference quotient for polynomial functions.

Application and Analysis (Analysis)

Calculate the limit of the difference quotient for polynomial functions.
Calculate the limit of the difference quotient for rational functions.

Evaluate and Create (Synthesis)

Determine using limits the slopes of lines tangent to rational functions at specified points on the function.
Determine the linear approximation a given rational function using the definition of the derivative as a slope.

Summary.

The notion of the derivative for general functions extends the notion of the derivative for polynomials and rational functions. To understand this more general notion and have a method for calculating derivatives of general functions, we first relate the difference quotient and its limit to the local linear approximation of polynomial and rational functions. In the next section, we will develop the idea of the local linear approximation for more general functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a sequence of partial sums.
Recognize the convergence of some common series.

Application and Analysis (Analysis)

Use the comparison test, alternating series test, and limit comparison test.

Evaluate and Create (Synthesis)

Analyze the exponential function as an infinite sum.

Summary.

This section introduces the notion of an infinite series and studies the convergence of such series. It also presents the exponential function as a differentiable function.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of rectifiable.
Recall the definition of the Riemann integral.

Application and Analysis (Analysis)

Use the Riemann integral to calculate areas.
Use series to approximate the arclength of a curve.
Use series and the area bounded by polygonal Jordan curves to approximate the area bounded by a Jordan Curve.
Use series to define the Riemann Integral.

Evaluate and Create (Synthesis)

Determine the area bounded by a Jordan curve defined by the graph of a function for some specific functions.
Contrast the simplified situation of calculating the area bounded by a Jordan curve defined by the graph of a continuous function with the area bounded by a general Jordan curve.

Summary.

In this section we develop the notion of the Riemann Integral and show that continuous functions are Riemann integrable over closed intervals. The introduction of the integral at this early stage follows the historical development of the subject as well as Courant’s approach in his calculus courses.

VI. Local Linear Approximation of Functions

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of the derivative of a function in terms of the local linear approximation of a function.

Recognize the connection between the limit of the difference quotient definition of the derivative and the definition in terms of local linear approximation.

Recall the basic differentiation rules for compound functions.

Application and Analysis (Analysis)

Calculate the derivatives of elementary functions.

Calculate the derivatives of compound functions using the basic rules of differentiation.

Approximate the value of a differentiable function near a known value.

Evaluate and Create (Synthesis)

Derive the basic differentiation rules using the definition of a derivative that comes from the local linear approximation of a function.

Approximate a root of a polynomial function that lies in a certain region using Newton's method.

Summary.

Rather than wait until the end of a course on differentiation to define the local linear approximation of a function as is usually done in calculus courses, we make this the primary tool for defining the derivative of a function. We pay careful attention to the error terms that are typically ignored in calculus courses. The basic rules and properties of differentiation are immediate consequences of the analysis of error terms that students already understand. Connecting the notion of a derivative of a general function to that of a polynomial function makes the geometric intuition underlying idea vivid. We introduce Newton's method at this point as well as the basic differentiation rules not only because it makes conceptual sense to do so, but because introducing these early on gives us a a greater variety of interesting examples to present as exercises for reinforcing learning of subsequent topics.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the derivatives of the basic (non-compound) elementary functions.
Identify useful ways to decompose a compound function for the sake of calculating its derivative.
Recognize the locus of points satisfying certain equations.
Recall the statement of the implicit function theorem in one variable.

Application and Analysis (Analysis)

Calculate the derivatives of elementary functions including roots, exponentials, logarithms, and the circular trigonometric functions.
Calculate the derivative of the inverse of a differentiable function.
Calculate the derivatives of compound functions using the basic rules of differentiation.
Approximate the value of a roots and of differentiable transcendental functions near a known value.
Calculate the derivatives of implicitly defined functions.
Calculate the lines tangent to curves given by the locus of point that solve certain equations.
Use logarithmic differentiation to differentiate functions.

Evaluate and Create (Synthesis)

Justify the critical estimates necessary for calculating the derivatives of the trigonometric functions.
Derive the formula for the derivative of an inverse of a differentiable function.
Justify the validity of the inverse function theorem using reflections and the local linear approximation of a function.
Model covarying quantities and linearize the equations that relate them.

Summary.

We carefully study the necessary estimates for determining the derivatives of the elementary functions. Students will master the basic calculation tools for calculating the derivatives

of elementary functions. As examples, they will use Newton's method to approximate roots of transcendental equations, calculate derivatives of inverse functions including the inverse trigonometric functions, and approximate the values of elementary functions near certain known values.The main theoretical tool in this section is the implicit function theorem. Students will learn how to use this theorem, its applications, and its limitations. This section is primarily a section on applications of differentiation to solve real world problems.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall that continuous functions attain their maximum and minimum values on closed and bounded intervals.
Recall Fermat's theorem, Rolles' Theorem, and the Mean Value Theorem.
Recall L'Hopital's Rule.
Identify limits that have indeterminate forms.

<u>Application and Analysis</u> (Analysis)

Calculate the antiderivatives of a function that is the derivative of a given function.
Calculate the extremal values of a differentiable function.
Calculate the minimum and maximum values of a differentiable function on a closed and bounded interval.
Use L'Hopital's rule calculate limits that have indeterminate forms.

<u>Evaluate and Create</u> (Synthesis)

Justify the fact that if the derivatives of two functions are equal on an interval, then the functions are equal up to a constant.
Develop mathematical models of physical systems to optimize measurable quantities.
Justify error estimates using the bounds on the derivatives of a function.

Summary.

This section explores the rigidity properties of differentiable functions. The mean value theorem establishes this rigidity. The mean value theorem will be critical to our proof of the fundamental theorem of calculus.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a second derivative.
Calculate second derivatives.
Determine if a function is twice differentiable at a point.

Application and Analysis (Analysis)

Analyze where a given function is increasing and decreasing from a sketch of the function.
Analyze from a sketch of a function where the functions slope is increasing.
Analyze inflection points of a function from a sketch of the function.

Evaluate and Create (Synthesis)

Approximate the change in slope using a second derivative.

Summary.

This section explores how information about the second derivative determines the shape of a function.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall what it means for a function to satisfy a (simple) first order differential equation.

Recognize a phase portrait for a differential equation.

Application and Analysis (Analysis)

Solve a simple differential equation using only the uniqueness of antiderivatives up to a constant.

Calculate the trajectories of ballistics.

Evaluate and Create (Synthesis)

Approximate the solution of a linear differential equation using successive local linear approximations.

Summary.

This section explores the connection between the local properties of a differentiable function and the global properties of the function. The mean value theorem establishes the fundamental relationship that is really a rigidity principle. The mean value theorem will be critical to our proof of the uniqueness of solutions of some simple differential equations.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition the derivative of a curve in the plane and in three dimensional space.

Recall that a state in a classical mechanical system is a position together with a velocity vector.

Describe parameterizations of standard surfaces and of any surface given by the graph of a function on the plane.

Application and Analysis (Analysis)

Calculate the velocity vectors associated to curves in the plane and in three dimensional space.

Calculate the velocity vectors associated to particles whose motion is restricted to a surface.

Determine the speed at which a particle moves as a function of time given its path of motion.

Evaluate and Create (Synthesis)

Generate examples of motions of particles restricted to surfaces.

Generate the equation for the plane to a surface at a point.

Build graphical representations for the tangent plane of a surface at a point.

Summary.

While usually taught only in multivariable calculus courses, the study of the kinematics of particle motion is purely a one dimensional problem. Studying this motion will help us to better understand the arc-length of curves as discussed later, gives us a host of interesting applications of calculus, and is necessary for our eventual solution of our motivating question of determining the area bounded by a Jordan curve. Of course, we will restrict our study to continuously differentiable Jordan curves in the final section of our course.

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