The Principles of Calculus I

Note: This course has already been constructed and is being currently implemented as the current Math 5. **Updated on 09-29-2021**

I. Decomposition

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the basic set operations.
Describe simple sets using set builder notation.

Application and Analysis (Analysis)

Calculate the intersections and unions of sets.
Determine which conditions are more or less restrictive.

Evaluate and Create (Synthesis)

Create examples of and counterexamples to conjectured statements about properties of unions and intersections.

Summary.

Students learn to use the language of sets and to use set builder notation to describe sets. Understanding the notion of a union and intersection of sets is critical for understanding decomposition. Intersection splits sets into pieces that can be reassembled by taking unions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall and identify the notation for intervals.
Match solutions of inequalities with intervals and intervals with corresponding inequalities.
Describe sets as unions of two intervals and intersections of two intervals. This is to be done both computationally and graphically.

Application and Analysis (Analysis)

Illustrate graphically the solutions of systems of inequalities as subsets of the line.
Calculate the intersection of unions of intervals.
Associate solutions of purely conjunctive or disjunctive systems of inequalities with intervals.
Calculate solutions of simple mixed (one conjunction and one disjunction) systems of inequalities in one variable.

Evaluate and Create (Synthesis)

Determine the solutions sets of complicated systems of inequalities.
Relate general statements about unions and intersections to unions and intersections of subsets of the line.

Summary.

Students practice basic arithmetic, manipulating equations, and manipulating inequalities. Students begin to solve problems by systematically breaking them up into simpler problems. Students improve their use of logical reasoning by working with conjunctions and disjunctions. This section prepares students for the next step, which is to work with piecewise defined functions.

• To Outline
• To Lectures

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a cartesian product and a relation.
Identify the components of a function: domain, range, natural domain co-domain. Note that natural domain is not standard terminology but is necessary since our functions are actually partial functions, a fact that should at least be mentioned.

Application and Analysis (Analysis)

Use the graphical representation of a function to determine its properties.
Determine where a function is positive, negative, and zero from its graphical depiction.
Determine intervals on which a function is increasing and decreasing from its graphical depiction.
Determine the local and global extrema of a function from its graphical depiction.

Evaluate and Create (Synthesis)

Compare the properties of two functions from graphical representations superimposed on the same coordinate grid.
Determine the domain and range of a function from its graphical representation.
Create a graphical representation of a function with specified properties.

Summary.

Students learn that functions are single valued subsets of a cartesian product. Although we specialize almost immediately to functions that are subsets of the plane, we do give examples of more general functions and students should periodically return to the more general setting for practice. Students learsn in this section how to obtain information from graphical representations of functions. This section prepares students to study functions given by explicit formulas.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the general form of linear and monomial functions.
Identify the parts of a linear function and a function given by a monic monomial: slope, y-intercept, and degree.
Identify the square root function and its graphical depiction.
Recognize the general appearance of a graph of the functions given above.

Application and Analysis (Analysis)

Evaluate at selected points a function given by a formula.
Determine the domain of a "simple" function given by a formula.
Calculate the slope of a line that passes through two given points.
Determine the equation of a line that passes through two points.
Calculate the x and y intercepts of a line that passes through two given points.

Evaluate and Create (Synthesis)

Calculate the range of a "simple" function with a domain that is given to be an interval that is not the entire real line.
Derive the various equations of a line using basic facts about similar triangles from euclidean geometry.

Summary.

This section introduces students to functions given by explicit formulas and teaches them how to evaluate functions. It provides them a foundation for further exploration. The geometrical discussion of lines is intended to develop their skill in using euclidean geometry and geometric intuition to determine the algebraic descriptions and properties of functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the meaning of restriction to subdomains and the basic operations with functions (sum, product, quotient, composition).

Identify the graph of a restricted function given the graph of the function. (Note: our functions are defined to be graphs and the language "graph of a function" just means "a graphical representation of the function".)

Application and Analysis (Analysis)

Calculate the iterated sums, products, quotients, and composites formed by multiple functions.

Break down a complicated function into simpler components in specified ways.

Evaluate and Create (Synthesis)

Determine the domain and range of some composite functions.

Summary.

Students learn how to construct compound functions and how to view complicated functions as being constructed from simpler functions. This is critical for them to later understand how transformations of the plane act on functions. It is also critical for them to understand later the various rules for differentiation.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Evaluate a piecewise defined function at given points.
Recognize what the partition is for any given piecewise defined function.
Describe the domain of a piecewise defined function.
Recall the meaning of a refinement of a partition.
Recall the meaning of commensurable partitions for two given piecewise defined functions.

Application and Analysis (Analysis)

Determine common refinements of two partitions.
Break down two given partitions for a piecewise defined function into a common refinement.
Use alternate presentations of two piecewise defined functions to compute their sum, product, and quotient.
Determine the range of a piecewise defined function (when reasonable).

Evaluate and Create (Synthesis)

Create functions with specified properties.
Determine the composite of two piecewise defined functions.
Generate the solution sets for equalities and inequalities involving piecewise defined functions.

Summary.

Students learn to work with piecewise defined functions by studying the properties of such functions on each of their defining regions. This section reinforces the students' understanding of the domain and range of a function. It also improves their grasp of conjunction and disjunction. The exercises they encounter are complicated and working through such exercises will help them to better understand the use of decomposition as a problem solving tool: to decompose complicated problems into simpler ones that are more easily solved.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Evaluate real valued functions of several variables at specified points in their domains.

Match graphical representations of (very basic) real valued functions of several variables with the sketches of their graphs.

Application and Analysis (Analysis)

Determine the level sets of a function on a planar domain.

Sketch the graph of a function defined on the boundary of a square.

Give original examples of functions of two variables and sketches of their graphs.

Evaluate and Create (Synthesis)

Sketch the graph of a function dependent on the distance on the square from fixed point on the square.

Determine the formula for a function dependent on the distance on the square from fixed point on the square.

Summary.

Students find it difficult to understand the trigonometric functions as functions on a circle. They also have difficulties in understanding the difference between the domain and range of a function. By introducing functions of several variables or functions on domains embedded in the plane, we provide students with geometric examples that more clearly distinguish between the domain and the range of a function. We also have students early on in the course study the analogs of trigonometric functions on, for example, the square. This helps them to better understand the idea of a function on a circle. The main difficulty is that our examples at this point are very limited because they do not yet know, for example, what a circle is or why a plane should have the equation it has because our tools are limited. In this first lesson, we give only very simple examples. Once students know more about vectors and circles, polynomial graphs, rotations, and so on, we can come up with more interesting examples. As we develop tools, we will have more interesting examples that we could not otherwise discuss without this section.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize if a point is a solution to a system of linear equations.
Recognize if a point is a solution to a system of linear inequalities.

Application and Analysis (Analysis)

Solve any system of linear equations by either repeated substitution or by ellimination.
Determine when a system of linear equations has a solution and does not have a solution.
Graphically determine the feasible set of a single linear inequality in two variables.

Evaluate and Create (Synthesis)

Graphically determine the feasible set of multiple linear inequalities in two variables.
Formulate in a set theoretic language the feasible set of multiple linear inequalities in two variables using a graphical realization of the feasible set.

Summary.

While this section is mostly review, we focus on teaching students to use graphical representation of data to discover complicated algebraic relationships. The approach further reinforces students' understanding of logical conjunction. They will intersect multiple feasible sets to determine the feasible set of a system and determine the boundary of the feasible set. They improve their understanding of set builder notation by carefully writing the feasible set in set builder notation.

II. Transformation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize how an arrow in the plane acts on points in the plane.
Explain what it means for a vector to be a set of equivalent arrows.
Recall that the difference between points in the plane is a vector, that vector can sum and admit multiplication by a real number, but that points in the plane do not sum.
Identify a coordinate representation for a vector.

Application and Analysis (Analysis)

Use the coordinate representation of a vector to translate points and sets.

Evaluate and Create (Synthesis)

Determine the action of vectors on functions.
Determine the equation for the translation of a locus of points in the plane that satisfies a given equation.

Summary.

Translation is the first rigid motion of the plane that we study. We study translation by studying vectors and the action of vectors on the plane. This is our main tool for studying all other rigid motions of the plane and for studying the parameterization of lines.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize the geometric meaning of scaling a vector.
Recall how to scale a vector given in coordinates.
Recall the definition of distance in the plane.

Application and Analysis (Analysis)

Determine the equation of a circle given specifying information.
Calculate the polar form of a vector.
Determine graphically the asymmetric axial scalings of subsets of the plane acting on subsets of the plane.
Calculate the way in which an asymmetric scaling of the plane acts on a function.

Evaluate and Create (Synthesis)

Determine the equation for a symmetric and an asymmetric axial scaling of a locus of points in the plane that satisfies a given equation.
Determine the equation for a general asymmetric non-axial scaling of a locus of points in the plane that satisfies a given equation.

Summary.

Scalings of the plane are transformations that are very important in the study of functions. The allow us to more easily sketch the graphs of functions, they give us a vivid way of understanding tangency, they help us to determining the asymptotic properties of functions. They also give us a rigorous way of understanding the scaling of physical objects, which is critical to the following section.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the fundamental units of a physical quantity.
Recognize if two physical quantities are commensurable.
Identify the conversion factor between commensurable physical quantities.
Describe a physical quantity using different units,

Application and Analysis (Analysis)

Solve linear scaling problems.
Solve simple nonlinear scaling problems.

Evaluate and Create (Synthesis)

Solve general nonlinear scaling problems.
Solve problems involving Galileo's square-cube law.

Summary.

Physical units arise up in any mathematical description of a physical experiment. We explore the both linear and nonlinear relationships in the scaling of physical quantities. We emphasize unit conversion and dimensional analysis. In settings where the relationships are not exact, we emphasize that the relationships we derive are not exact. For example, the change in the volume of paint on a sphere where the thickness of the paint remains the same but the radius changes is a complex relationship, however, the volume approximately changes as the surface area so long as the layer of paint is thin. The students should understand that we are investigating an approximation in these cases. Studying units will help students with applications and in deriving relationships between variables in application problems.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the vector equation for a line.
Recall the form of a parameterization of a line.
Identify the vector that translates points along a given line given either two points on the line or the slope of the line.

Application and Analysis (Analysis)

Determine the position of a point on a line segment with information about absolute distance from another point on the line.
Determine the position of a point on a line segment with information about relative distance from other points on the line.
Determine from a transformational perspective the position of a particle that is at a point at one time and another point at another time.
Determine from a transformational perspective the equation of motion of a particle that moves on a given line in a certain direction at a certain speed.
Use vectors to find the midpoint of a line segment or a point that divides a line segment into segments with specified ratios of lengths.

Evaluate and Create (Synthesis)

Evaluate whether or not two particles in linear motion collide.
Formulate the equation of motion of a particle that moves in a piecewise linear fashion.
Determine the equation for a locus of points in the plane that is the translation of a locus of points satisfying a given equation.

Summary.

This section covers the basic ideas needed to understand and describe linear motion. This is a critical section in the flow of ideas in the course and provides students will the basic tools that they will need in many later sections, including the next. Students will need to be very fluent with their computational skills.

• To Outline
• To Lectures

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the expression for a vector that is perpendicular to another vector.
Interpret the relative orientations of two orthogonal vectors.
Recall the slope of a line perpendicular to another line.
Recognize graphically the set of points that is the reflection across a line of a different collection of points.

Application and Analysis (Analysis)

Determine the equation of a line that is perpendicular to a given line.
Calculate the position of a point on a line that is closest to a point in the plane.
Create rectangles in the plane with specified side lengths or ratios of side lengths.
Formulate as a piecewise defined function in time the equation of motion of a particle moving on a rectangle in the plane.

Evaluate and Create (Synthesis)

Determine the reflection of a point in the plane across a given line.
Determine the equation for a locus of points in the plane that is the reflection of a locus of points satisfying a given equation.

Summary.

The main point of this section is for students to understand the meaning of reflection and how to compute the reflection of a point across a line. A critical application is in the next section, where they study inverse functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall that the reflection of the point (a, b) across the line given by $y = x$ is the point (b, a).

Recognize graphically why the horizontal line test for a planar function is the vertical line test for its reflection across $y = x$.

Recall the condition for invertibility of a function in a general setting.

Explain the relationship between the general condition for invertibility and its graphical interpretation in the planar setting in terms of reflection.

Application and Analysis (Analysis)

Calculate the formula for the inverse of some simple functions.

Determine points on the inverse function that correspond to specified points on the function.

Determine the set theoretic inverse of a function given a function that has certain specified points or that is given by a simple formula.

Use restriction to make a non-invertible function invertible on a smaller domains.

Evaluate and Create (Synthesis)

Create schemes to estimate the values of the inverse function at specified points given a function that is easy to evaluate.

Calculate the inverse of a function whose domain has been restricted to guarantee the function’s invertibility.

Summary.

We use reflection to define and determine the inverse of a planar function and extend these definitions to more general settings of non-planar functions. The geometric perspective will help us later in studying tangency to inverse functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize the graphical interpretation of adding points on a circle.
Recall the group addition law for points on the unit circle in cartesian coordinates.
Recall the formula for rotation of points around the origin by a given angle.

Application and Analysis (Analysis)

Calculate the rotation of a point about an arbitrary point.

Evaluate and Create (Synthesis)

Determine the equation for a locus of points in the plane that is the rotation about a given point of a locus of points satisfying a given equation.

Summary.

We use basic ideas about translation and perpendicularity to derive the group addition law on the unit circle. We use scaling and translation to extend the group addition law to define rotation about arbitrary points in the plane. Measurement of angles and the angle summation formulas of the trigonometric functions will follow immediately from our understanding of rotation.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Recall the definition of a fraction of a circle.
- Recall the interpretation of an angle between rays as a point on the unit circle.
- Recall the meaning of a measurement of an angle.
- Match the sine, cosine, and tangent functions with their geometric realizations as y-coordinate, x-coordinate, and slope.
- Recall the meaning of angle of elevation and angle of depression in surveying problems.

Application and Analysis (Analysis)

- Determine the angle summation formulas from the group operation on the circle.
- Determine the Pythagorean identities from the defining equation for the unit circle.
- Solve for all points on the unit circle with a single specified coordinate.
- Calculate parameterizations of paths given by the motion of a particle on a circle with specified constant speed.
- Use graphical information to determine the angles associated to various points on a circle given information about the angles given by related points.
- Use polar coordinates and convert from polar to cartesian coordinates.

Evaluate and Create (Synthesis)

- Derive the half angle formulas.
- Create a scheme for approximating the location in cartesian coordinates of points on a circle with certain angle measures.
- Solve complicated surveying problems.

Summary.

Starting from an understanding of rotation and the group structure of a circle, we develop a basic understanding of the trigonometric functions. We will later explore inverse trigonometric functions, periodicity, waves, and so on.

• To Outline
• To Lectures

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Interpret graphically rotation by half of a circle.
Interpret graphically reflection across the axes.
Interpret graphically the inversion of the y-axis by the map $y \mapsto \frac{1}{y}$ for all (x, y) with $y \neq 0$.

Application and Analysis (Analysis)

Determine the action of the above three involutions on planar functions.

Evaluate and Create (Synthesis)

Sketch functions determined by the reciprocals of monomials using inversion of the y-axis.

Summary.

The current section puts us in a position to study rational functions by showing students how to use transformations to sketch reciprocals of monomials. It also sets the stage for studying symmetries of a function like evenness and oddness.

III. Rigidity

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall that lines are determined by two points.
Identify parameterizations of lines in three dimensions.
Identify equations for planes.
Recall the dot product of two vectors.

Application and Analysis (Analysis)

Determine the intersection of lines and planes.
Parameterize the intersection of planes in three dimensions.
Use the dot product to determine perpendicularity.

Evaluate and Create (Synthesis)

Determine planes by a normal vector and a point.

Summary.

Determination of lines and planes as well as determination of intersections of lines and planes give us our first examples of the utility of rigidity principles.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Match quadratic functions with their graphs.
Identify which quadratic polynomials have a maximum and which have a minimum.
Recall the quadratic formula.
Recall the factor theorem.
Identify the remainder of a polynomial on division by a linear polynomial.
Identify the zeros and the orders of zeros of a polynomial function in factored form.
Identify the asymptotic behavior of a polynomial.

Application and Analysis (Analysis)

Determine the maximum possible zeros and minimum possible zeros of a polynomial of a given degree.
Apply the factor theorem to long division.
Use transformations to derive the quadratic formula.

Evaluate and Create (Synthesis)

Sketch the graph of a polynomial function using local and global data.
Determine the solution to optimization problems involving quadratic functions: how close do boats in linear motion get to each other, what is the closest distance from a point to a line, maximize area bounded by a polygon of fixed perimeter, etc.

Summary.

This section covers polynomial functions from the point of view of transformation and rigidity. The factor theorem is really a rigidity result because of the uniqueness of the decomposition. Up to a scaling of the y-axis, a polynomial written in factored form is really determined by its zeros and the degrees of its zeros. This affords us a simple way of sketching polynomials in factored form. Later, we will develop a notion of tangency by studying the intersection of lines with polynomials. That viewpoint will be essential in later understanding tangency in the transcendental setting.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the zeros and polls (and their orders) of a rational function whose numerator and denominator are written in factored form.
Identify the asymptotic behavior of a rational function.

Application and Analysis (Analysis)

Present a rational function as a polynomial plus a rational function in proper form.
Sketch the reciprocal of a factored polynomial using inversion of the y-axis.

Evaluate and Create (Synthesis)

Use the local and global properties of a rational function given by a quotient of two polynomials in factored form to sketch the function.
Use the sketch of a rational function to determine its properties.

Summary.

This section studies rational functions by using the rigidity of such functions and their local and global behavior to graphically analyze them.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify commensurable partitions for piecewise rational functions.

Application and Analysis (Analysis)

Use the graphs of polynomial and rational functions to determine the solutions to inequalities.
Calculate the sum, product, and quotient of piecewise rational functions.
Use the graphs of polynomial and rational functions to determine the solutions to compound inequalities.

Evaluate and Create (Synthesis)

Compose piecewise rational functions.
Solve inequalities involving piecewise rational functions using graphical methods.

Summary.

This section emphasizes the power of spatial reasoning in solving complicated inequalities involving piecewise rational functions. It builds students' ability to use spatial reasoning to solve computationally difficult problems.

IV. Symmetry

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Identify the symmetries of various sets.
Recall the definition of an odd and even function.
Identify even and odd functions from their graphical representations.
Identify the transformations under which even and odd functions are symmetric.

<u>Application and Analysis</u> (Analysis)

Determine the symmetry groups associated to different sets.
Demonstrate computationally whether a function is even or odd.

<u>Evaluate and Create</u> (Synthesis)

Argue that a given function is neither even nor odd.

Summary.

This section explores the symmetry of functions under reflection across the y-axis (even functions) and under rotation by half a circle (odd functions). It gives students an introduction to symmetry and the idea of a symmetry group.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a period of a functions and a periodic function.
Recognize periodicity and periods from the sketch of a function.
Match the trigonometric functions with graphical representations of subsets of the plane.
Identify the domains where trigonometric functions are invertible.
Identify the intersections of a horizontal line with a periodic function given certain principle intersections.

Application and Analysis (Analysis)

Determine the graphical representation of reciprocal trigonometric functions using y-axis inversion.
Determine the graphical representation of sinusoidal functions.
Calculate composites of trigonometric and inverse trigonometric functions.

Evaluate and Create (Synthesis)

Solve equations involving trigonometric functions.
Generate simulations of traveling waves.
Present a superposition of two sinusoidal functions as a varying amplitude multiplying a sinusoidal function.

Summary.

This section goes more in depth into the solution of trigonometric equations and the graphical representation of trigonometric functions. It also studies the application of trigonometric functions to the description of waves.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify whether a function satisfies a linear or exponential model of change.
Recall the various properties of exponential functions and logarithms.
Recall the definition of half-life, growth rate, and decay rate.
Identify the graphical representations of exponentials and logarithms.

Application and Analysis (Analysis)

Calculate simple logarithms.
Solve simple exponential and logarithmic equations.

Evaluate and Create (Synthesis)

Formulate models of change given data that follows an exponential or linear model of change.
Calculate half-line (or any "fractional life" or doubling life, etc.) of a function satisfying an exponential growth model.
Determine future values of a function satisfying linear or exponential growth models given two values at different times. This is an application of a rigidity principle.

Summary.

We study exponential functions and logarithms from the point of view that exponential growth satisfies a certain symmetry in growth, they change by the same factor on time intervals of the same length. This symmetry condition imposes a rigidity condition on the class of such functions, namely, they are determined by their values at two different time points. Use of transformational principles obviates most calculations and dramatically simplifies solving the majority of applied problems.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of tangency for polynomials.
Recall the definition of tangency for rational functions
Recall the power rule for taking derivatives from an algebraic perspective.

Application and Analysis (Analysis)

Determine the equations of lines that are tangent to polynomial curves at given points.
Determine the slope of lines that are tangent by employing basic rules.
Determine the equations of lines that are tangent to rational curves at given points.

Evaluate and Create (Synthesis)

Determine the reflections of lines off of planar curves.
Find all points on a curve where the tangency is greater than second degree.

Summary.

In the setting of polynomial functions, tangency is purely algebraic. Namely, a line L is tangent to a polynomial function f at $(a, f(a))$ if $f - L$ is a polynomial with a degree two or greater intersection at a. This makes tangency in the algebraic setting a very geometrically vivid concept. Tangency is really about a certain kind of asymmetry of the intersection under increasingly large symmetric scalings of the plane. A line L is tangent to a rational function f at $(a, f(a))$ if $f - L$ is a rational function whose numerator has a degree two or greater zero at a.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of tangency for roots.
Interpret the idea of tangency to an inverse function using symmetry under reflection.
Recall that tangential intersections are degree two intersections except for at most finitely many points.

Application and Analysis (Analysis)

Determine the equations of lines that are tangent to certain inverse functions, powers and rational functions, at given points.
Calculate points where a polynomial attains an extremal value.

Evaluate and Create (Synthesis)

Determine the reflections of light rays off mirrors that have the shape of rational functions and roots.

Summary.

We use a reflection symmetry of tangency to determine the lines tangent to roots. This allows us to determine the lines tangent to functions even when functions may have cusps and are not differentiable in the analytical sense. We explore this notion of tangency and see that there are only finitely many points on f where tangency is greater than a degree two tangency. In the next course, we will show how this notion of tangency can be extended to transcendental functions.

The Principles of Calculus I

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