The Principles of Calculus III
Sample Final 1

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n}{2^n}$ convergent or divergent?

(b) Use the fact that

$$n^2 + 3n + 2 = (n+1)(n+2)$$

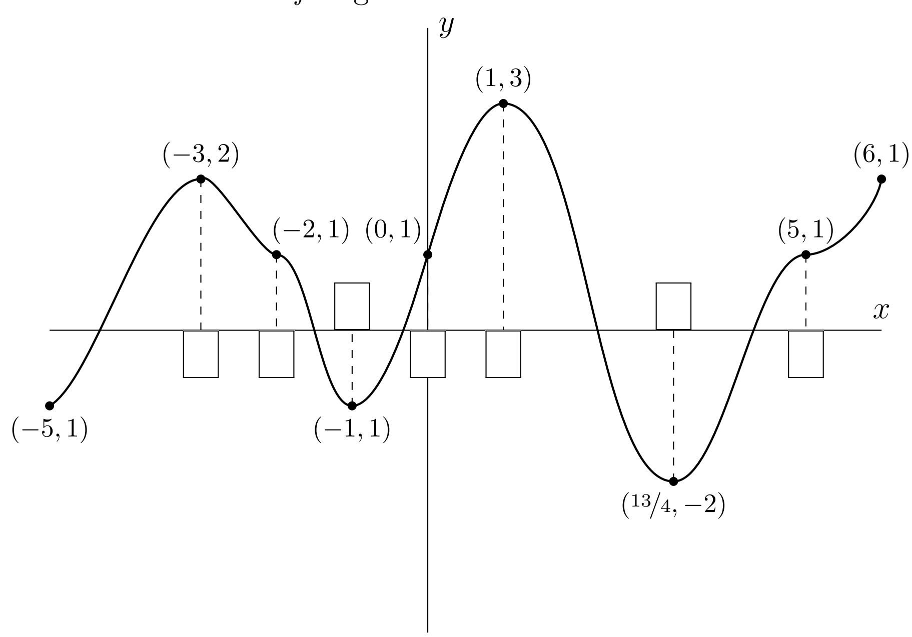
to calculate $\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n}}{n!}$?

Problem C3.

A sketch of the function f is given below.



Put the appropriate symbols $+$, $-$, or 0 in the given boxes to indicate the value of the second derivative at the x value of the given point in f.

Problem C4.

Calculate the Maclaurin series for f where

$$f(x) = 2x\sin(x^2).$$

Problem C5.

Solve the initial value problem

$$\begin{cases} f''(x) = 3x^2 - 1 \\ f'(1) = 2 \\ f(3) = 5. \end{cases}$$

Problem C6.

A particle moves along the segment of the line that passes through $(1, 2, 5)$ and $(3, 7, 12)$. At time $t = 0$, it is at $(1, 2, 5)$. The particle always moves to the right at a speed s, where for each non-negative real number t the function s is given by the equation

$$s(t) = 1 + t^2.$$

Find an equation for the position of the particle at time t.

Problem C7.

Calculate the average value of the function f on $[4, 9]$, where

$$f(x) = \frac{1}{x - 2}.$$

Problem C8.

Determine whether or not the following integrals are convergent or divergent. Explain your reasoning.

(a) $\displaystyle\int_0^\infty \frac{1}{x^2 + 2x + 2}\, \mathrm{d}x$

(b) $\displaystyle\int_1^\infty \frac{\sin(x)}{\sqrt{x^3}}\, \mathrm{d}x$

(c) $\displaystyle\int_0^\infty \frac{x}{x^2 + \mathrm{e}^{-x}}\, \mathrm{d}x$

(d) $\displaystyle\int_1^3 \frac{x\mathrm{e}^x}{x - 1}\, \mathrm{d}x$

(e) $\displaystyle\int_1^4 \frac{x^2 + 2}{\sqrt{x - 1}}\, \mathrm{d}x$

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\int_1^4 \sqrt{2+\sqrt{x}}\,\mathrm{d}x$;

(b) $\int_1^5 x\ln(x)\,\mathrm{d}x$;

Problem C10.

Interpret the following integrals as areas of regions, sketch the region, and use either basic geometry or transformations to calculate the integral.

(a)

$$\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x =$$

(b)

$$\int_0^1 x\,\mathrm{d}x =$$

(c) If f is continuous and invertible, $f(0)=0$, $f(1)=5$, and $\int_0^1 f(x)\,\mathrm{d}x = 2$, then

$$\int_0^5 f^{-1}(x)\,\mathrm{d}x =$$

Problem C11.

A particle moves in $\mathbb{R}^3$ and its position is given for each real number t by $c(t)$, where

$$c(t) = (t^2, \sin(t), 2t-5).$$

Set up but do not solve an integral that describes the length of the path that the particle traverses between time 0 and time 3.

Problem C12.

Use a transformation to turn the following integral into an integral of a rational function:

$$\int \frac{\sin(x)+\cos(x)}{1+\cos^2(x)}\,\mathrm{d}x.$$

Set up but do not evaluate integral that results from performing this substitution.

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B Level Questions

Problem B1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (2t, t^2)$.

(a) Calculate the velocity vector of c at time $t = 1$ and its acceleration vector.

(b) What is the component of the acceleration vector in the direction of motion?

(c) What is the instantaneous rate of change of the speed of c at time t?

(d) What is the magnitude of the acceleration vector at time t?

Problem B2.

Solve the initial value problem

$$\begin{cases} y' = (2 - y)(1 + y) \\ y(0) = c. \end{cases}$$

Find a formula for y and then calculate $\lim_{x \to \infty} y(x)$ when $c = 1$ and when $c = 4$.

Problem B3.

The curves given by $y = (x - 1)^2$ and $y = x + 2$ bound a region R in the plane. Rotate R about the y-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let S be the surface given by $S = \{(x, y, x^2 + y^3) \colon (x, y) \in \mathbb{R}^2\}$. A particle of mass m moves along the surface so that the x–y coordinates are given for all t by $(2t - 1, t + 1)$.

(a) Find the equation of the plane that is tangent to S at the point $(1, 2, 9)$.

(b) What is the force on the particle in the direction of the upward pointing normal when the particle is at the point $(1, 2, 9)$?

(c) Set up (but do not evaluate) an integral that describes the length of the path that the particle traverses during the time interval $[1, 2]$.

(c) How much work does the surface do on the particle during the time interval $[1, 2]$?

Problem A2.

You control the motion of a point particle that moves in the plane. It has initial position $(0, 0)$ and initial velocity $\langle 10, -2\rangle$, where units of distance given in feet and units of time are given in seconds. The particle is equipped with accelerometers. The accelerometers give readings that can only be guaranteed to be accurate to $\pm\frac{1}{10}$ feet per square second. According to the accelerometers, the acceleration of the particle at time t is given as $a(t) = \langle 2t+1, t+\sin(t)\rangle$.

(a) According to the accelerometer data, where should the particle be at time t?

(b) Let $P(t)$ be the set of possible locations of the particle at time t. Given the error in the accelerometers, determine $P(t)$.

(c) What is the area of $P(t)$ as a function of time?

(d) The particle must not deviate by more than 10 feet from its predicted location or it will be impossible to accurately control. How often must precise location and velocity measurements be taken to ensure that accurate control is possible?

Problem A3.

Suppose that f has at least four continuous derivatives. Calculate $\int_0^1 f(x)\,\mathrm{d}x$ in the following way. Take an even partition of $[0, 1]$ with n steps. For each interval in the partition, integrate the first three terms of the Taylor expansion for f centered at the interval's midpoint.

(a) What is the area given by your approximation for each interval in the partition?

(b) Write down a formula for this approximation method. Be sure to simplify as much as possible so that you obtain more usable formula. Be sure to look for and eliminate any terms that are guaranteed to not contribute to the value of the integral.

(c) If the maximum value of the fourth derivative of f is M, then what is the maximum possible error in the approximation of the integral?

The Principles of Calculus III

Sample Final 2

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C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n-1}{3n^2-2}$ convergent or divergent?

(b) Calculate $\sum_{n=1}^{\infty} \frac{1+3^n}{4^{n-1}}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$? Check for convergence at the endpoints.

Problem C3.

Suppose that g is given for each real number x by

$$g(x) = 2 + \frac{1}{3}x + \frac{1}{10}x^2 - \frac{3}{20}x^3 + O(x^4).$$

Find the first three terms of the Maclaurin series for f where

$$f(x) = g(x)e^{2x}.$$

Problem C4.

Suppose that f is at least four times differentiable and that

$$f(2) = 3, \; f'(2) = 1, \;\; f''(2) = 5, \text{ and } f'''(2) = 7.$$

Calculate the first four terms of the Taylor series for f centered at $x = 2$.

Problem C5.

Suppose that f and g are analytic functions on $\mathbb{R}$ and that for each natural number n,

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2} + g\left(\frac{1}{n}\right).$$

Given that $g(3) = 5$, calculate $f(3)$.

Problem C6.

A particle moves in the plane with velocity vector v given for each non-negative real t by

$$v(t) = \langle 2t + 1, 3t^2 \rangle.$$

When t is 3, the position of the particle is $(1, 5)$. What is the initial position of the particle (position at time $t = 0$)?

Problem C7.

Estimate $\int_2^{10} \ln(x)\,\mathrm{d}x$ using the midpoint rule with four intervals. Be sure to include in your estimate a bound on the error.

Problem C8.

Calculate $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_{\sin(x)}^{x^2+1} \mathrm{e}^{s^2}\,\mathrm{d}s.$

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\displaystyle\int_0^1 x\cos(4 - x^2)\,\mathrm{d}x;$

(b) $\displaystyle\int_0^{\frac{\pi}{12}} x\sin(3x)\,\mathrm{d}x.$

Problem C10.

Use partial fractions to write the integral $\displaystyle\int \frac{1}{(x-2)(2x^2+3)(x^2+1)^2}\,\mathrm{d}x$ in a simplified form. Do not solve for the undetermined coefficients.

Problem C11.

Let f be the function given for each real number x by $f(x) = 4\sin(x)$. Calculate the length of the arc determined by the segment of f between $(0, 0)$ and $(\pi, 0)$.

Problem C12.

Use Green's theorem to calculate the area determined by $\displaystyle\int_0^1 x^2\,\mathrm{d}x$. Verify that your answer is correct.

B Level Questions

Problem B1.

A particle moves along the skin of the cone C with vertex at $(0,0,0)$ and whose projection onto the plane $x=0$ is the set of points given by $z=|y|$. The particle makes a complete counter clockwise (when viewed form above) rotation around the cone one time each second and the vertical component of the particle's velocity vector is $\langle 0,0,2t\rangle$.

(a) What is the particle's velocity vector at time t?

(b) What is the particle's acceleration vector at time t?

(c) Set up but do not evaluate an integral that describes the distance the particle has traveled between time $t=0$ and time $t=2$.

Problem B2.

The curves given by $y=x^2$ and $y=x^3$ form a simple closed curve that bounds a region R in the plane. Set up but do not evaluate any integral used in this problem.

(a) What is the length of the curve?

(b) Determine the area bounded by R without using Green's theorem.

(c) Use Green's theorem to determine the area bounded by R.

Problem B3.

The curves given by $y=x^2+1$ and $y=x+3$ bound a region R in the plane. Rotate R about the x-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 4 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (t, 2t + 1)$. Imagine that P is a very thin, stationary, and nondeformable sheet and that a particle whose motion is determined by c is moving along and under P.

(a) When does the particle experience the greatest normal force?

(b) Where is the particle at this time?

Problem A2.

Let f be the function given by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 4. \end{cases}$$

Find a sufficiently small mesh size for an even partition for $[0, 4]$ so that you can approximate $\int_0^4 f(x)\,\mathrm{d}x$ to within $\frac{1}{10}$ of the actual value of the integral.

(a) What is a reasonable mesh size?

(b) What is the approximate value that you have calculated?

(c) What is the actual value of the Riemann Integral of the given function on $[0, 4]$.?

Problem A3.

Suppose that S is the surface given by

$$S = \{(x, y, z) \colon (x, y) \in \mathbb{R}^2,\ z = |y|\}.$$

If c is a differentiable curve on S given by $c(t) = (x(t), y(t), z(t))$, then what are the possible velocity vectors for c when $y = 0$?