The Principles of Calculus II
Sample Final 1

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

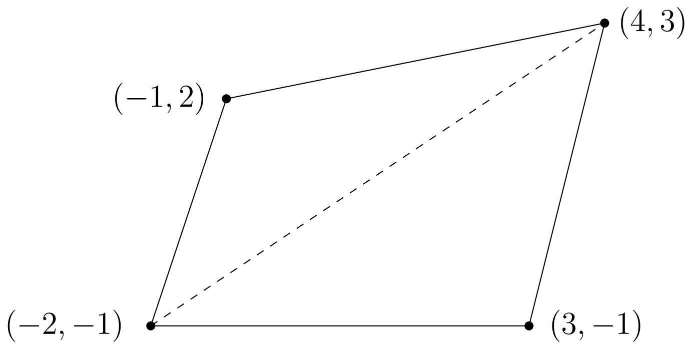
The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

Use the shoelace formula for the area of a triangle to determine the area of the given polygon.



The line segment $\overline{(-2,-1)(4,3)}$ divides the polygon above into two triangles. What is the altitude and area of each triangle?

Problem C2.

(a) Let $a_n = \frac{2n^3+n+2}{n^3-5}$. Calculate $\lim\limits_{n\to\infty} a_n$.

(b) Suppose that (a_n) is a sequence that is convergent to 0. Calculate $\lim_{n\to\infty} \frac{\sin(5a_n)}{3a_n}$.

Problem C3.

(a) Let f be the function defined by

$$f(x) = \begin{cases} 2x + a & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$$

Find a so that f is continuous.

(b) Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x < a \\ 4x - 4 & \text{if } x \geq a. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\lim\limits_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$.

(b) Calculate $\lim_{x\to\infty} \dfrac{3x^2+2x-1}{x^2+7}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = 2x - 1 + o(x^2) \quad \text{and} \quad g(x) = x + o(x).$$

(a) Calculate $(fg)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

(b) Calculate $(f \circ g)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

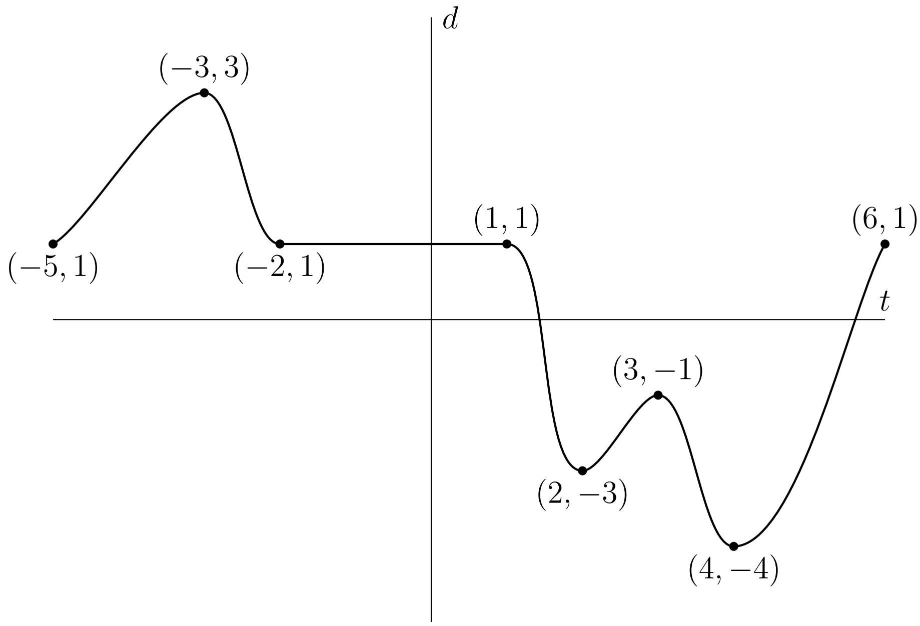
Problem C6.

(a) Calculate $\lim_{x\to 2^+} \dfrac{|x-2|}{x^2-4}$.

(b) Calculate $\lim_{x\to 2^-} \dfrac{|x-3|}{x^2-9}$.

Problem C7.

Below is a sketch of the position of a particle with respect to time. Measurements of the particle's position are taken on the time interval $[-5, 6]$.



(a) When is the particle at rest?

(b) When is particle moving in the positive direction?

(c) When is the particle moving in the negative direction?

Problem C8.

At time t, the position of a particle is given by γ, where

$$\gamma(t) = (t^2, t^3 + 1).$$

What is the velocity vector of the particle at time 5?

Problem C9.

Calculate the following limits:

(a) $\lim_{x \to 4^+} \frac{x^2 - 16}{\ln(x-4)}$;

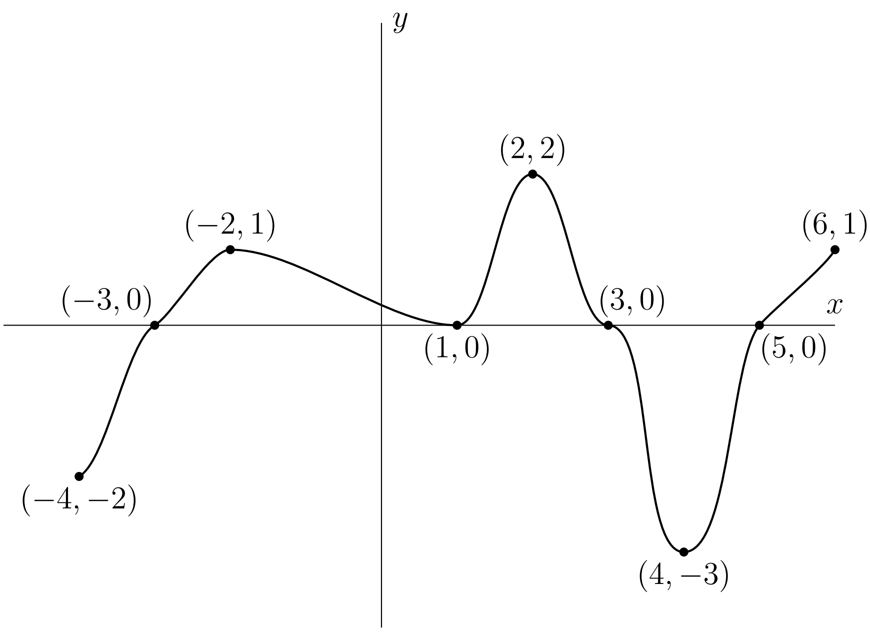
(b) $\lim_{x \to 1^+} (9x - 9)^{\frac{1}{x-1}}$.

Problem C10.

Use the definition of the derivative to approximate the value of $\log_2(5)$.

Problem C11.

Suppose that f is a differentiable function and that this is a sketch of the graph of f':



List the extremal values of f in $(-4, 6)$ and determine if they are local maxima or local minima.

Problem C12.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying $z = 9 - x^2 - y^2$. A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (1+t, 2+t^2) \quad \text{with} \quad -\infty < t < \infty.$$

(a) What is the velocity vector of the particle at time $t = 0$?

(b) When $t = 0$, how fast is the particle moving?

B Level Questions

Problem B1.

Given the sequences (a_n) below, calculate $\lim_{n\to\infty} a_n$ and briefly justify your calculations:

(a) $a_n = 5^{\frac{1}{n}}$;

(b) $a_n = (5^n + 3^n)^{\frac{1}{n}}$.

Problem B2.

Three cubic inches per second of air is being pumped into a spherical balloon. How fast is the surface area changing when the radius is 12 inches?

Problem B3.

Show that the function f given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

is differentiable at 0. Calculate $f'(0)$.

A Level Questions

Problem A1.

The product fg is defined on all of $\mathbb{R}$ except possibly at finitely many points, differentiable where it is defined, defined at 0, and

$$f'(x)g(x) = x^2 - f(x)g'(x).$$

Given that

$$g(x) = 2x + 1 \quad \text{and} \quad f(0)g(0) = 1,$$

find a candidate for $f(x)$. Is your answer unique? Provide an explanation.

Problem A2.

A point particle is at the point $(1, 4)$ initially and travels with a constant velocity to the point $(3, 2)$, where it strikes the curve given by $y^3 = x^2 - 1$. It reflects off of the curve and continues onward indefinitely. The particle is always moving at unit speed. Describe the position of the particle as a function of time.

Problem A3.

A function f is differentiable on the interval $(1, 4)$ and $f(2) = 10$. The maximum value of $|f'|$ on $(1, 4)$ is 7. Find a positive real number E so that the values of f on $(2 - E, 2 + E)$ are guaranteed to lie in the interval $(8, 12)$.

The Principles of Calculus II
Sample Final 2

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C Level Questions

Problem C1.

Let A be the region bounded by the curves $y = 0$, $x = 2$, $x = 10$, and $y = \ln(x)$. Use a Riemann sum approximation with an even partition with four intervals and a midpoint tagging to approximate the area of A.

Problem C2.

Suppose that $a_1 = 1$ and that $a_{n+1} = \sqrt{6 + a_n}$. Calculate $\lim_{n\to\infty} a_n$. You may use the fact that the square root function is continuous, but be sure to justify that the limit exists.

Problem C3.

Let f be the function defined by

$$f(x) = \begin{cases} a & \text{if } x \leq 0 \\ \frac{\sin(2x)}{5x} & \text{if } x > 0. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\displaystyle\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$.

(b) Calculate $\displaystyle\lim_{x\to\infty} \frac{x^3 + x^2 - 1}{2 - 3x^3}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = x + 1 + o(x - 2) \quad \text{and} \quad g(x) = 5x + o(x - 2).$$

Calculate $\displaystyle\lim_{x\to 2} \frac{(f \circ g)(x) - (f \circ g)(2)}{x - 2}$ using the appropriate notation.

Problem C6.

Suppose that f is given by

$$f(x)=\begin{cases}\frac{x+2}{x+1} & \text{if } x\in(-\infty,-1)\cup(-1,0)\\ 0 & \text{if } x=0\\ \frac{\sin(2x)}{x} & \text{if } x\in(0,\infty).\end{cases}$$

Calculate $\lim_{x\to 0} f(x)$.

Problem C7.

Suppose that $f(2)=3$, $g(2)=5$, $f'(2)=3$, and $g'(2)=4$. Suppose further that

$$h(x)=(f(x))^2+f(x)\sqrt{g(x)}.$$

Calculate $h'(2)$.

Problem C8.

A particle moves counterclockwise along a circular track of radius 5. It makes one complete revolution every 3 seconds. What is its velocity vector at time $t=1$?

Problem C9.

Calculate the following limits:

(a) $\lim_{x\to 0^+} x\log_2(x)$;

(b) $\lim_{x\to 0^+} x^{2x}$.

Problem C10.

A function f is differentiable on all of $\mathbb{R}$ and has the property that if x is a real number, then

$$f'(x)=2x+1.$$

If $f(0)=2$, what is $f(x)$?

Problem C11.

The function f increases to the left of -2 and has a local maximum when $x=-2$. It decreases on $(-2,1)$ and has a local minimum at $x=1$. It increases on $(1,3)$ and has another local maximum when $x=3$. The function is decreasing on $(3,\infty)$. Sketch a function that can potentially correspond to f'.

Problem C12.

Let P be the plane given by the locus of points in $\mathbb{R}^3$ satisfying

$$z - 2x + 3y = 5.$$

A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (t^2, t^3 + t - 1) \quad \text{with} \quad 0 \leq t \leq 4.$$

(a) What is the velocity vector of the particle at time $t = 1$?

(b) When $t = 1$, how fast is the particle moving upwards?

B Level Questions

Problem B1.

Calculate $\lim_{n\to\infty} \sqrt[3]{a+\frac{1}{n}}$. Justify your solution.

Problem B2.

Use Newton's method to estimate $\sqrt{7}$ by applying the method to a suitable quadratic polynomial. Stop at the third iteration of the method. Draw a picture that graphically illustrates what you are doing.

Problem B3.

For every time t, the velocity vector of a particle is given by

$$\gamma'(t) = \langle 1, t - 3t^2 \rangle.$$

At time 0, the particle is at the position $(0, 1)$. Where is the particle when $t = 2$? Use the appropriate theorems to justify your work.

A Level Questions

Problem A1.

Show that there is exactly one solution to the initial value problem

$$\begin{cases} y'(x) = 5y \\ y(0) = 1. \end{cases}$$

Hint: You will need to take the quotient of two solutions.

Problem A2.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Find an equation of the plane that is tangent to P at $(1, 2, 4)$.

Problem A3.

Suppose that the function f is differentiable and that if x is a real number then $f'(x)$ is in $[-6, 2]$. Estimate $f(3 + h)$ if $f(3)$ is equal to 10.