The Principles of Calculus I

Sample Final 1

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Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

The line L passes through the points $(2,3)$ and $(5,9)$.

(a) Find an equation for L.

(b) If $L_\perp$ is perpendicular to L, then what is the slope of $L_\perp$?

Problem C2.

Recall that 3 feet is one yard and 60 seconds is one minute. Calculate the acceleration $7\,\frac{\mathrm{ft}}{\mathrm{s}^2}$ in units of yards and minutes.

Problem C3.

Suppose that a line segment L passes through the points $(1,3)$ and $(2,5)$.

(a) Find the point on L that is a distance of 2 from the point $(1,3)$ and that lies to the right of $(1,3)$.

(b) Find the point on the line segment L whose distance from $(1,3)$ is one third the length of the line segment L.

Problem C4.

Suppose that L is a line of slope 5 that intersects the origin. Where does L intersect the unit circle?

Problem C5.

Take f to be the function given by

$$f(x) = |x+1|.$$

Write the function f explicitly as a piecewise defined function.

Problem C6.

Rotate the point $(2,3)$ by the angle $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ around the point $(1,5)$.

Problem C7.

Take f to be the function given by

$$f(x) = x^2 - 4x + 7.$$

What is the minimum y value for a point in f?

Problem C8.

Take f to be the function given by

$$f(x) = (x+2)^2(x-1)(x-4)^3.$$

Sketch f and then find all x with $f(x) \geq 0$.

Problem C9.

What is the leading term of the polynomial f given by

$$f(x) = (-2x+1)^3(3x+1)^2(x-8)^4(x-1)^{100}?$$

For some constant C and some natural number n, your answer should look like Cx^n.

Problem C10.

What fraction of a circle is an arc of $\frac{5}{3}$ radians? A Zuma-radian is a measure of an angle. There are 120 Zuma-radians in a circle. How many Zuma-radians is $\frac{5}{3}$ radians?

Problem C11.

Suppose that A and B are angles in the first quadrant and $A > B$. Put the correct symbol, $<$ or $>$, in the boxes below.

$\cos(A)$	$\square$	$\cos(B)$
$\sin(A)$	$\square$	$\sin(B)$
$\tan(A)$	$\square$	$\tan(B)$

Problem C12.

Suppose that $\log_2(A) = 3$, $\log_2(B) = 2$, and $\log_2(C) = 4$. Calculate $\log_2\left(\frac{AC^2}{B^5}\right)$.

B Level Questions

Problem B1.

Compute $(f+g)(x)$, where f and g are given by

$$f(x) = \begin{cases} 2x+5 & \text{if } x \leq -1 \\ 3x & \text{if } x > -1 \end{cases} \quad \text{and} \quad g(x) = |x+3|.$$

Problem B2.

There is a building in front of you. The angle of elevation from your position to the top of the building is 10°. You walk 50 feet towards the building and measure the angle of elevation to now be 25°. How tall is the building?

Problem B3.

(a) It takes 2 workers 5 hours to shovel 100 cubic feet of sand. How many hours does it take 7 workers to shovel 300 cubic feet of sand?

(b) A pyramid has a height of 5 feet and a surface area of 50 square feet. A larger pyramid with the same proportions has a height of 12 feet. What is its surface area?

(c) A pyramid has a height of 6 feet and a volume of 10 cubic feet. If a smaller pyramid of the same proportions has a volume of 2 cubic feet, what is its height?

A Level Questions

Problem A1.

At time zero, Boat A is initially at $(0, 0)$ and Boat B is initially at $(1, 4)$. Boat A has velocity vector $\langle 1, 3\rangle$ and Boat B has velocity vector $\langle -1, 2\rangle$. When are the boats closest to eachother and how far apart are they at this time? Be sure to include the proper units in your answer.

Problem A2.

Let L_1 be the line given by $y = 2x$ and let L_2 be the line given by $y = 3x + 1$. Find an equation of the line given by the reflection of L_2 across L_1. To clarify: L_1 is the "mirror" and L_2 is being reflected.

Problem A3.

Find all solutions to the inequality $|x^2 - 4| > 2$.

Principles of Calculus I
Sample Final 2

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C Level Questions

Problem C1.

Suppose that L_1 is the line given by

$$y = -x + 7$$

and L_2 is the line given by

$$y = 4x - 3.$$

Where do L_1 and L_2 intersect?

Problem C2.

Sketch on a real number line the set $(-7, 9] \cap \Big([-10, 2] \cup (5, 10)\Big)$.

Problem C3.

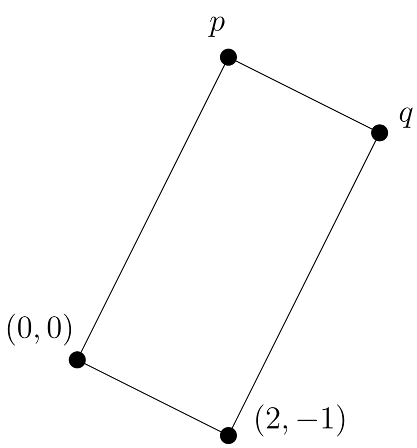
Let L be the line segment with endpoints $(1, 5)$ and $(6, 15)$. The point p lies on L. The distance from $(1, 5)$ to p is one fifth the distance of p to $(6, 15)$. What are the coordinates of p?

Problem C4.

The vector $\langle 2, 3\rangle$ translates points along the line L. The point $(1, 4)$ lies on L. Find the coordinates of the point on L that lies to the right of $(1, 4)$ and that is a distance of 7 from $(1, 4)$.

Problem C5.

The shape below is a rectangle, whose short side is half the length of the long side. Calculate the coordinates for p, q and write an equation for the line passing through p and q.



Problem C6.

Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } -2 < x \leq 1 \\ 3x - 2 & \text{if } x > 3. \end{cases}$$

What is the domain of f and the range of f?

Problem C7.

Suppose that f and g are given by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ 3x^2 & \text{if } 0 \leq x \leq 5 \\ 6x & \text{if } x > 5 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \cos(x) & \text{if } x < 2 \\ x^4 + 1 & \text{if } x \geq 2. \end{cases}$$

Write $f + g$ as a piecewise defined function.

Problem C8.

Find all solutions to the equation

$$(2\sin(3\theta) - 1)(4\cos(5\theta) + 1)(\sin(\theta) + 5) = 0.$$

Problem C9.

(a) Suppose that that $A(0) = 7$ and $A(1) = 3$. If A changes exponentially with respect to time, find a formula for $A(t)$ at any time t.

(b) Suppose that that $A(2) = 7$ and $A(5) = 3$. If A changes exponentially with respect to time, find a formula for $A(t)$ at any time t.

Problem C10.

Take f to be the polynomial given by

$$f(x) = (x + 5)^7(x + 2)(x - 3)^4.$$

Sketch f and find all x with $f(x) \leq 0$.

Problem C11.

(a) Where does the line $y = 10x$ intersect the unit circle?

(b) $\cos(\tan^{-1}(10)) =$

(c) $\sin(\tan^{-1}(10)) =$

Problem C12.

Calculate:

(a) $\arccos\left(\sin\left(-\frac{\pi}{7}\right)\right)$;

(b) $\arcsin\left(\sin\left(\frac{4\pi}{9}\right)\right)$.

B Level Questions

Problem B1.

Write the feasible set of the system

$$\begin{cases} x + y \leq 1 \\ y - 2x < 4 \\ y - x \geq 1 \end{cases}$$

in set builder notation.

Problem B2.

(a) Take f to be the rational function given by

$$f(x) = \frac{(x+5)^9(x+3)^4(x-2)^3}{(x+7)^3(x-1)^2(x-6)^4}.$$

Sketch f. Be sure to emphasize the global and local behavior of f and to NOT sketch f to scale.

(b) Let h be the function given by $h(x) = |f(x)|$. Write h as a piecewise defined function.

Problem B3.

A function A is changing exponentially with respect to time. It models the decay of a certain radioactive substance. If $A(0) = 12$ and $A(5) = 7$ then what is the half-life of the substance? Units of time are measured here in hours.

A Level Questions

Problem A1.

Let f be the function defined by the formula

$$f(x) = x^3 - 4x + 1.$$

What is an equation of the line tangent to the graph of f at the point $(2, 1)$? Be sure to directly use the algebraic definition of tangency—you will earn **no credit** for using other techniques.

Problem A2.

Suppose that f and g are the functions given by

$$f(x) = \begin{cases} 5x^2 & \text{if } x < 1 \\ 2x & \text{if } x > 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 5 - 2x & \text{if } x < 3 \\ x - 1 & \text{if } x \geq 6. \end{cases}$$

Write $f \circ g$ as a piecewise defined function.

Problem A3.

You swim to a buoy and then back. You can swim at a speed of 2 miles per hour with respect to still water. There is a current moving from the direction of the buoy to the shore at a speed s. When you are swimming against the current, you will naturally swim more slowly with respect to the stationary buoy, but when you swim back to shore, you will move more quickly with respect to the buoy. The buoy is a distance of 2 miles from the shore. For what value of s will your total swimming time be minimal? Does your answer depend on the buoy's distance or on your speed?

The Principles of Calculus II
Sample Final 1

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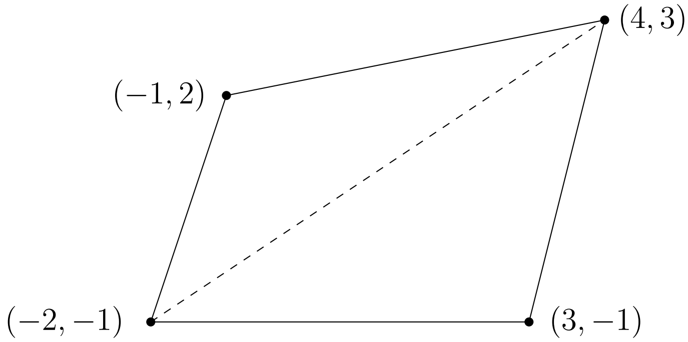
C Level Questions

Problem C1.

Use the shoelace formula for the area of a triangle to determine the area of the given polygon.



The line segment $\overline{(-2,-1)(4,3)}$ divides the polygon above into two triangles. What is the altitude and area of each triangle?

Problem C2.

(a) Let $a_n = \frac{2n^3+n+2}{n^3-5}$. Calculate $\lim\limits_{n\to\infty} a_n$.

(b) Suppose that (a_n) is a sequence that is convergent to 0. Calculate $\lim_{n\to\infty} \frac{\sin(5a_n)}{3a_n}$.

Problem C3.

(a) Let f be the function defined by

$$f(x) = \begin{cases} 2x+a & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$$

Find a so that f is continuous.

(b) Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x < a \\ 4x-4 & \text{if } x \geq a. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\lim\limits_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$.

(b) Calculate $\lim_{x\to\infty} \dfrac{3x^2+2x-1}{x^2+7}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = 2x - 1 + o(x^2) \quad \text{and} \quad g(x) = x + o(x).$$

(a) Calculate $(fg)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

(b) Calculate $(f \circ g)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

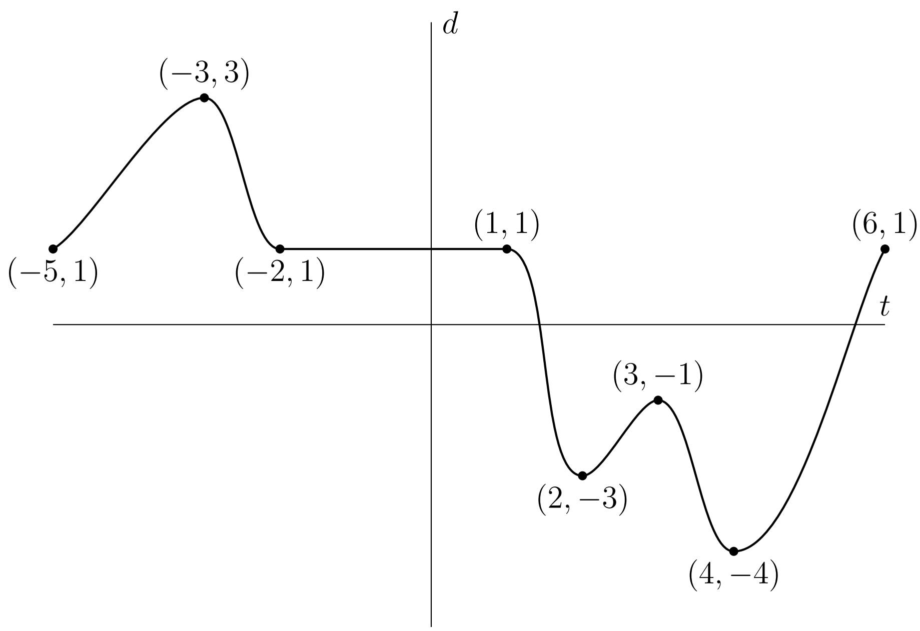
Problem C6.

(a) Calculate $\lim_{x\to 2^+} \dfrac{|x-2|}{x^2-4}$.

(b) Calculate $\lim_{x\to 2^-} \dfrac{|x-3|}{x^2-9}$.

Problem C7.

Below is a sketch of the position of a particle with respect to time. Measurements of the particle's position are taken on the time interval $[-5, 6]$.



(a) When is the particle at rest?

(b) When is particle moving in the positive direction?

(c) When is the particle moving in the negative direction?

Problem C8.

At time t, the position of a particle is given by γ, where

$$\gamma(t) = (t^2, t^3 + 1).$$

What is the velocity vector of the particle at time 5?

Problem C9.

Calculate the following limits:

(a) $\lim_{x \to 4^+} \frac{x^2 - 16}{\ln(x - 4)}$;

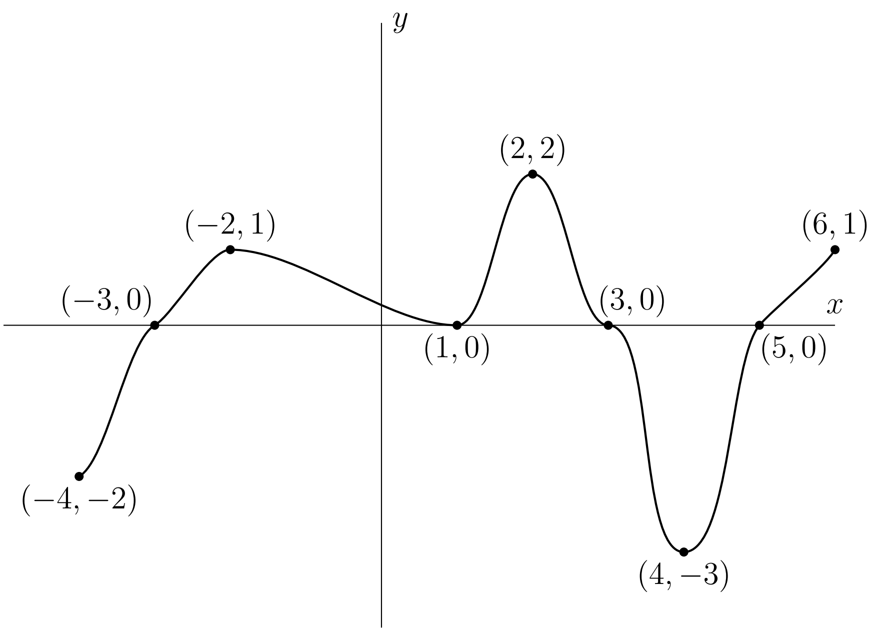
(b) $\lim_{x \to 1^+} (9x - 9)^{\frac{1}{x-1}}$.

Problem C10.

Use the definition of the derivative to approximate the value of $\log_2(5)$.

Problem C11.

Suppose that f is a differentiable function and that this is a sketch of the graph of f':



List the extremal values of f in $(-4, 6)$ and determine if they are local maxima or local minima.

Problem C12.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying $z = 9 - x^2 - y^2$. A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (1 + t, 2 + t^2) \quad \text{with} \quad -\infty < t < \infty.$$

(a) What is the velocity vector of the particle at time $t = 0$?

(b) When $t = 0$, how fast is the particle moving?

B Level Questions

Problem B1.

Given the sequences (a_n) below, calculate $\lim_{n\to\infty} a_n$ and briefly justify your calculations:

(a) $a_n = 5^{\frac{1}{n}}$;

(b) $a_n = (5^n + 3^n)^{\frac{1}{n}}$.

Problem B2.

Three cubic inches per second of air is being pumped into a spherical balloon. How fast is the surface area changing when the radius is 12 inches?

Problem B3.

Show that the function f given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

is differentiable at 0. Calculate $f'(0)$.

A Level Questions

Problem A1.

The product fg is defined on all of $\mathbb{R}$ except possibly at finitely many points, differentiable where it is defined, defined at 0, and

$$f'(x)g(x) = x^2 - f(x)g'(x).$$

Given that

$$g(x) = 2x + 1 \quad \text{and} \quad f(0)g(0) = 1,$$

find a candidate for $f(x)$. Is your answer unique? Provide an explanation.

Problem A2.

A point particle is at the point $(1, 4)$ initially and travels with a constant velocity to the point $(3, 2)$, where it strikes the curve given by $y^3 = x^2 - 1$. It reflects off of the curve and continues onward indefinitely. The particle is always moving at unit speed. Describe the position of the particle as a function of time.

Problem A3.

A function f is differentiable on the interval $(1, 4)$ and $f(2) = 10$. The maximum value of $|f'|$ on $(1, 4)$ is 7. Find a positive real number E so that the values of f on $(2 - E, 2 + E)$ are guaranteed to lie in the interval $(8, 12)$.

The Principles of Calculus II

Sample Final 2

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C Level Questions

Problem C1.

Let A be the region bounded by the curves $y = 0$, $x = 2$, $x = 10$, and $y = \ln(x)$. Use a Riemann sum approximation with an even partition with four intervals and a midpoint tagging to approximate the area of A.

Problem C2.

Suppose that $a_1 = 1$ and that $a_{n+1} = \sqrt{6 + a_n}$. Calculate $\lim_{n\to\infty} a_n$. You may use the fact that the square root function is continuous, but be sure to justify that the limit exists.

Problem C3.

Let f be the function defined by

$$f(x) = \begin{cases} a & \text{if } x \leq 0 \\ \frac{\sin(2x)}{5x} & \text{if } x > 0. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$.

(b) Calculate $\lim_{x\to\infty} \dfrac{x^3 + x^2 - 1}{2 - 3x^3}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = x + 1 + o(x - 2) \quad \text{and} \quad g(x) = 5x + o(x - 2).$$

Calculate $\lim_{x\to 2} \dfrac{(f \circ g)(x) - (f \circ g)(2)}{x - 2}$ using the appropriate notation.

Problem C6.

Suppose that f is given by

$$f(x) = \begin{cases} \frac{x+2}{x+1} & \text{if } x \in (-\infty, -1) \cup (-1, 0) \\ 0 & \text{if } x = 0 \\ \frac{\sin(2x)}{x} & \text{if } x \in (0, \infty). \end{cases}$$

Calculate $\lim_{x \to 0} f(x)$.

Problem C7.

Suppose that $f(2) = 3$, $g(2) = 5$, $f'(2) = 3$, and $g'(2) = 4$. Suppose further that

$$h(x) = (f(x))^2 + f(x)\sqrt{g(x)}.$$

Calculate $h'(2)$.

Problem C8.

A particle moves counterclockwise along a circular track of radius 5. It makes one complete revolution every 3 seconds. What is its velocity vector at time $t = 1$?

Problem C9.

Calculate the following limits:

(a) $\lim_{x \to 0^+} x \log_2(x)$;

(b) $\lim_{x \to 0^+} x^{2x}$.

Problem C10.

A function f is differentiable on all of $\mathbb{R}$ and has the property that if x is a real number, then

$$f'(x) = 2x + 1.$$

If $f(0) = 2$, what is $f(x)$?

Problem C11.

The function f increases to the left of -2 and has a local maximum when $x = -2$. It decreases on $(-2, 1)$ and has a local minimum at $x = 1$. It increases on $(1, 3)$ and has another local maximum when $x = 3$. The function is decreasing on $(3, \infty)$. Sketch a function that can potentially correspond to f'.

Problem C12.

Let P be the plane given by the locus of points in $\mathbb{R}^3$ satisfying

$$z - 2x + 3y = 5.$$

A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (t^2, t^3 + t - 1) \quad \text{with} \quad 0 \leq t \leq 4.$$

(a) What is the velocity vector of the particle at time $t = 1$?

(b) When $t = 1$, how fast is the particle moving upwards?

B Level Questions

Problem B1.

Calculate $\lim_{n\to\infty} \sqrt[3]{a+\frac{1}{n}}$. Justify your solution.

Problem B2.

Use Newton's method to estimate $\sqrt{7}$ by applying the method to a suitable quadratic polynomial. Stop at the third iteration of the method. Draw a picture that graphically illustrates what you are doing.

Problem B3.

For every time t, the velocity vector of a particle is given by

$$\gamma'(t) = \langle 1, t - 3t^2 \rangle.$$

At time 0, the particle is at the position $(0, 1)$. Where is the particle when $t = 2$? Use the appropriate theorems to justify your work.

A Level Questions

Problem A1.

Show that there is exactly one solution to the initial value problem

$$\begin{cases} y'(x) = 5y \\ y(0) = 1. \end{cases}$$

Hint: You will need to take the quotient of two solutions.

Problem A2.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Find an equation of the plane that is tangent to P at $(1, 2, 4)$.

Problem A3.

Suppose that the function f is differentiable and that if x is a real number then $f'(x)$ is in $[-6, 2]$. Estimate $f(3 + h)$ if $f(3)$ is equal to 10.

The Principles of Calculus III
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C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n}{2^n}$ convergent or divergent?

(b) Use the fact that

$$n^2 + 3n + 2 = (n+1)(n+2)$$

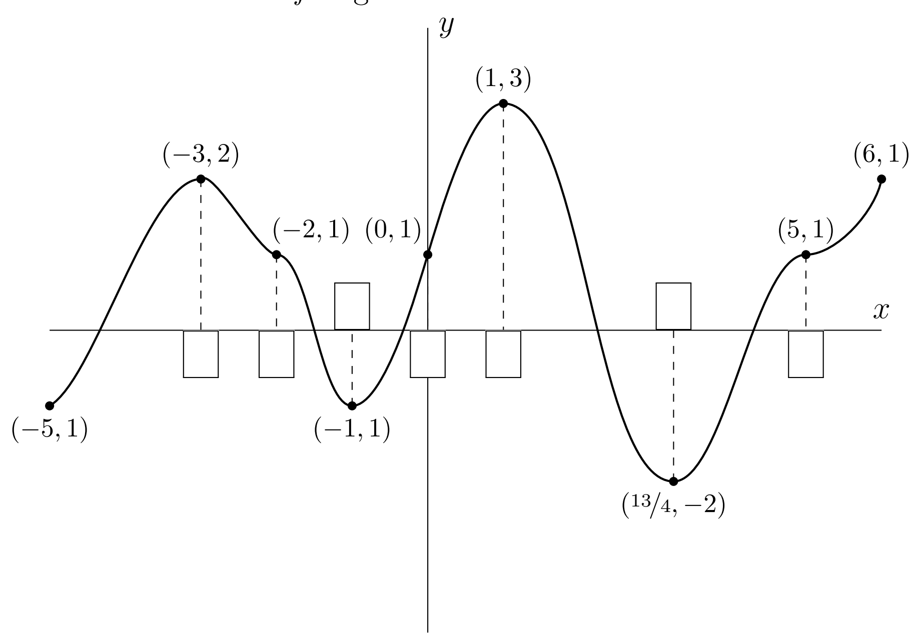
to calculate $\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n}}{n!}$?

Problem C3.

A sketch of the function f is given below.



Put the appropriate symbols +, −, or 0 in the given boxes to indicate the value of the second derivative at the x value of the given point in f.

Problem C4.

Calculate the Maclaurin series for f where

$$f(x) = 2x \sin(x^2).$$

Problem C5.

Solve the initial value problem

$$\begin{cases} f''(x) = 3x^2 - 1 \\ f'(1) = 2 \\ f(3) = 5. \end{cases}$$

Problem C6.

A particle moves along the segment of the line that passes through $(1, 2, 5)$ and $(3, 7, 12)$. At time $t = 0$, it is at $(1, 2, 5)$. The particle always moves to the right at a speed s, where for each non-negative real number t the function s is given by the equation

$$s(t) = 1 + t^2.$$

Find an equation for the position of the particle at time t.

Problem C7.

Calculate the average value of the function f on $[4, 9]$, where

$$f(x) = \frac{1}{x - 2}.$$

Problem C8.

Determine whether or not the following integrals are convergent or divergent. Explain your reasoning.

(a) $\displaystyle\int_0^\infty \frac{1}{x^2 + 2x + 2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^\infty \frac{\sin(x)}{\sqrt{x^3}}\,\mathrm{d}x$

(c) $\displaystyle\int_0^\infty \frac{x}{x^2 + \mathrm{e}^{-x}}\,\mathrm{d}x$

(d) $\displaystyle\int_1^3 \frac{x\mathrm{e}^x}{x - 1}\,\mathrm{d}x$

(e) $\displaystyle\int_1^4 \frac{x^2 + 2}{\sqrt{x - 1}}\,\mathrm{d}x$

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\int_1^4 \sqrt{2+\sqrt{x}}\,\mathrm{d}x$;

(b) $\int_1^5 x\ln(x)\,\mathrm{d}x$;

Problem C10.

Interpret the following integrals as areas of regions, sketch the region, and use either basic geometry or transformations to calculate the integral.

(a)

$$\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x =$$

(b)

$$\int_0^1 x\,\mathrm{d}x =$$

(c) If f is continuous and invertible, $f(0)=0$, $f(1)=5$, and $\int_0^1 f(x)\,\mathrm{d}x = 2$, then

$$\int_0^5 f^{-1}(x)\,\mathrm{d}x =$$

Problem C11.

A particle moves in $\mathbb{R}^3$ and its position is given for each real number t by $c(t)$, where

$$c(t) = (t^2, \sin(t), 2t-5).$$

Set up but do not solve an integral that describes the length of the path that the particle traverses between time 0 and time 3.

Problem C12.

Use a transformation to turn the following integral into an integral of a rational function:

$$\int \frac{\sin(x)+\cos(x)}{1+\cos^2(x)}\,\mathrm{d}x.$$

Set up but do not evaluate integral that results from performing this substitution.

B Level Questions

Problem B1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (2t, t^2)$.

(a) Calculate the velocity vector of c at time $t = 1$ and its acceleration vector.

(b) What is the component of the acceleration vector in the direction of motion?

(c) What is the instantaneous rate of change of the speed of c at time t?

(d) What is the magnitude of the acceleration vector at time t?

Problem B2.

Solve the initial value problem

$$\begin{cases} y' = (2 - y)(1 + y) \\ y(0) = c. \end{cases}$$

Find a formula for y and then calculate $\lim_{x \to \infty} y(x)$ when $c = 1$ and when $c = 4$.

Problem B3.

The curves given by $y = (x - 1)^2$ and $y = x + 2$ bound a region R in the plane. Rotate R about the y-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let S be the surface given by $S = \{(x, y, x^2 + y^3) \colon (x, y) \in \mathbb{R}^2\}$. A particle of mass m moves along the surface so that the x–y coordinates are given for all t by $(2t - 1, t + 1)$.

(a) Find the equation of the plane that is tangent to S at the point $(1, 2, 9)$.

(b) What is the force on the particle in the direction of the upward pointing normal when the particle is at the point $(1, 2, 9)$?

(c) Set up (but do not evaluate) an integral that describes the length of the path that the particle traverses during the time interval $[1, 2]$.

(c) How much work does the surface do on the particle during the time interval $[1, 2]$?

Problem A2.

You control the motion of a point particle that moves in the plane. It has initial position $(0, 0)$ and initial velocity $\langle 10, -2\rangle$, where units of distance given in feet and units of time are given in seconds. The particle is equipped with accelerometers. The accelerometers give readings that can only be guaranteed to be accurate to $\pm\frac{1}{10}$ feet per square second. According to the accelerometers, the acceleration of the particle at time t is given as $a(t) = \langle 2t + 1, t + \sin(t)\rangle$.

(a) According to the accelerometer data, where should the particle be at time t?

(b) Let $P(t)$ be the set of possible locations of the particle at time t. Given the error in the accelerometers, determine $P(t)$.

(c) What is the area of $P(t)$ as a function of time?

(d) The particle must not deviate by more than 10 feet from its predicted location or it will be impossible to accurately control. How often must precise location and velocity measurements be taken to ensure that accurate control is possible?

Problem A3.

Suppose that f has at least four continuous derivatives. Calculate $\int_0^1 f(x)\,\mathrm{d}x$ in the following way. Take an even partition of $[0, 1]$ with n steps. For each interval in the partition, integrate the first three terms of the Taylor expansion for f centered at the interval's midpoint.

(a) What is the area given by your approximation for each interval in the partition?

(b) Write down a formula for this approximation method. Be sure to simplify as much as possible so that you obtain more usable formula. Be sure to look for and eliminate any terms that are guaranteed to not contribute to the value of the integral.

(c) If the maximum value of the fourth derivative of f is M, then what is the maximum possible error in the approximation of the integral?

The Principles of Calculus III
Sample Final 2

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n-1}{3n^2-2}$ convergent or divergent?

(b) Calculate $\sum_{n=1}^{\infty} \frac{1+3^n}{4^{n-1}}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$? Check for convergence at the endpoints.

Problem C3.

Suppose that g is given for each real number x by

$$g(x) = 2 + \frac{1}{3}x + \frac{1}{10}x^2 - \frac{3}{20}x^3 + O(x^4).$$

Find the first three terms of the Maclaurin series for f where

$$f(x) = g(x)e^{2x}.$$

Problem C4.

Suppose that f is at least four times differentiable and that

$$f(2) = 3,\ f'(2) = 1,\ \ f''(2) = 5,\ \text{and}\ f'''(2) = 7.$$

Calculate the first four terms of the Taylor series for f centered at $x = 2$.

Problem C5.

Suppose that f and g are analytic functions on $\mathbb{R}$ and that for each natural number n,

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2} + g\left(\frac{1}{n}\right).$$

Given that $g(3) = 5$, calculate $f(3)$.

Problem C6.

A particle moves in the plane with velocity vector v given for each non-negative real t by

$$v(t) = \langle 2t+1, 3t^2 \rangle.$$

When t is 3, the position of the particle is $(1,5)$. What is the initial position of the particle (position at time $t=0$)?

Problem C7.

Estimate $\int_2^{10} \ln(x)\,\mathrm{d}x$ using the midpoint rule with four intervals. Be sure to include in your estimate a bound on the error.

Problem C8.

Calculate $\dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_{\sin(x)}^{x^2+1} \mathrm{e}^{s^2}\,\mathrm{d}s$.

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\displaystyle\int_0^1 x\cos(4-x^2)\,\mathrm{d}x$;

(b) $\displaystyle\int_0^{\frac{\pi}{12}} x\sin(3x)\,\mathrm{d}x$.

Problem C10.

Use partial fractions to write the integral $\displaystyle\int \frac{1}{(x-2)(2x^2+3)(x^2+1)^2}\,\mathrm{d}x$ in a simplified form. Do not solve for the undetermined coefficients.

Problem C11.

Let f be the function given for each real number x by $f(x) = 4\sin(x)$. Calculate the length of the arc determined by the segment of f between $(0,0)$ and $(\pi,0)$.

Problem C12.

Use Green's theorem to calculate the area determined by $\displaystyle\int_0^1 x^2\,\mathrm{d}x$. Verify that your answer is correct.

B Level Questions

Problem B1.

A particle moves along the skin of the cone C with vertex at $(0, 0, 0)$ and whose projection onto the plane $x = 0$ is the set of points given by $z = |y|$. The particle makes a complete counter clockwise (when viewed form above) rotation around the cone one time each second and the vertical component of the particle's velocity vector is $\langle 0, 0, 2t\rangle$.

(a) What is the particle's velocity vector at time t?

(b) What is the particle's acceleration vector at time t?

(c) Set up but do not evaluate an integral that describes the distance the particle has traveled between time $t = 0$ and time $t = 2$.

Problem B2.

The curves given by $y = x^2$ and $y = x^3$ form a simple closed curve that bounds a region R in the plane. Set up but do not evaluate any integral used in this problem.

(a) What is the length of the curve?

(b) Determine the area bounded by R without using Green's theorem.

(c) Use Green's theorem to determine the area bounded by R.

Problem B3.

The curves given by $y = x^2 + 1$ and $y = x + 3$ bound a region R in the plane. Rotate R about the x-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 4 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (t, 2t + 1)$. Imagine that P is a very thin, stationary, and nondeformable sheet and that a particle whose motion is determined by c is moving along and under P.

(a) When does the particle experience the greatest normal force?

(b) Where is the particle at this time?

Problem A2.

Let f be the function given by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 2 \\ x - 2 & \text{if } 2 \le x \le 4. \end{cases}$$

Find a sufficiently small mesh size for an even partition for $[0, 4]$ so that you can approximate $\int_0^4 f(x)\,\mathrm{d}x$ to within $\frac{1}{10}$ of the actual value of the integral.

(a) What is a reasonable mesh size?

(b) What is the approximate value that you have calculated?

(c) What is the actual value of the Riemann Integral of the given function on $[0, 4]$.?

Problem A3.

Suppose that S is the surface given by

$$S = \{(x, y, z) \colon (x, y) \in \mathbb{R}^2,\ z = |y|\}.$$

If c is a differentiable curve on S given by $c(t) = (x(t), y(t), z(t))$, then what are the possible velocity vectors for c when $y = 0$?