Course Outline – Common Features

Common Features

Lecture: three hours per week.

Discussion sections: two hours per week. One is a collaborative learning workshop. One is a directed question and answer period.

Homework: has two parts. The first part consists of self-check problems delivered electronically. The second part consists of careful write ups of the worksheets that have previously undergone guided peer review.

Exams: One midterm and one final.

Video: Video content supplements discussion section content and should be watched prior to discussion section.

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Course Outline

The Principles of Calculus I

Note: This course has already been constructed and is being currently implemented.

- I. Decomposition
- II. Transformation
- III. Rigidity
- IV. Symmetry

Short Description:

The first course in the three course sequence, The Principles of Calculus I is a course on the application of transformation and symmetry to the study of elementary functions. Decomposition, Transformation, Rigidity, and Symmetry are the four themes of the course. Topics include: functions and their graphical representation; operations on and with functions; set builder notation; linear systems; the group of isometries of the plane; scalings of the plane; physical units; trigonometry and its applications; inverse functions; inversion of the y-axis; polynomial and rational functions; rigidity of rational functions; rigidity and intersections of lines and planes; graphical representations of trigonometric functions and sinusoidal functions; trigonometric equations; exponential functions; logarithms; tangency in an algebraic setting.

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Principles of Calculus I

Note: This course has already been constructed and is being currently implemented as the current Math 5.

• II. Transformation

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● III. Rigidity

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● IV. Symmetry

The Principles of Calculus I

Note: This course has already been constructed and is being currently implemented as the current Math 5 at UC Riverside.

I. Decomposition

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the basic set operations.
Describe simple sets using set builder notation.

Application and Analysis (Analysis)

Calculate the intersections and unions of sets.
Determine which conditions are more or less restrictive.

Evaluate and Create (Synthesis)

Create examples of and counterexamples to conjectured statements about properties of unions and intersections.

Summary.

Students learn to use the language of sets and to use set builder notation to describe sets. Understanding the notion of a union and intersection of sets is critical for understanding decomposition. Intersection splits sets into pieces that can be reassembled by taking unions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Recall and identify the notation for intervals.
- Match solutions of inequalities with intervals and intervals with corresponding inequalities.
- Describe sets as unions of two intervals and intersections of two intervals. This is to be done both computationally and graphically.

Application and Analysis (Analysis)

- Illustrate graphically the solutions of systems of inequalities as subsets of the line.
- Calculate the intersection of unions of intervals.
- Associate solutions of purely conjunctive or disjunctive systems of inequalities with intervals.
- Calculate solutions of simple mixed (one conjunction and one disjunction) systems of inequalities in one variable.

Evaluate and Create (Synthesis)

- Determine the solutions sets of complicated systems of inequalities.
- Relate general statements about unions and intersections to unions and intersections of subsets of the line.

Summary.

Students practice basic arithmetic, manipulating equations, and manipulating inequalities. Students begin to solve problems by systematically breaking them up into simpler problems. Students improve their use of logical reasoning by working with conjunctions and disjunctions. This section prepares students for the next step, which is to work with piecewise defined functions.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall the definition of a cartesian product and a relation.
Identify the components of a function: domain, range, natural domain co-domain. Note that natural domain is not standard terminology but is necessary since our functions are actually partial functions, a fact that should at least be mentioned.

<u>Application and Analysis</u> (Analysis)

Use the graphical representation of a function to determine its properties.
Determine where a function is positive, negative, and zero from its graphical depiction.
Determine intervals on which a function is increasing and decreasing from its graphical depiction.
Determine the local and global extrema of a function from its graphical depiction.

<u>Evaluate and Create</u> (Synthesis)

Compare the properties of two functions from graphical representations superimposed on the same coordinate grid.
Determine the domain and range of a function from its graphical representation.
Create a graphical representation of a function with specified properties.

Summary.

Students learn that functions are single valued subsets of a cartesian product. Although we specialize almost immediately to functions that are subsets of the plane, we do give examples of more general functions and students should periodically return to the more general setting for practice. Students learsn in this section how to obtain information from graphical representations of functions. This section prepares students to study functions given by explicit formulas.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the general form of linear and monomial functions.
Identify the parts of a linear function and a function given by a monic monomial: slope, y-intercept, and degree.
Identify the square root function and its graphical depiction.
Recognize the general appearance of a graph of the functions given above.

Application and Analysis (Analysis)

Evaluate at selected points a function given by a formula.
Determine the domain of a "simple" function given by a formula.
Calculate the slope of a line that passes through two given points.
Determine the equation of a line that passes through two points.
Calculate the x and y intercepts of a line that passes through two given points.

Evaluate and Create (Synthesis)

Calculate the range of a "simple" function with a domain that is given to be an interval that is not the entire real line.
Derive the various equations of a line using basic facts about similar triangles from euclidean geometry.

Summary.

This section introduces students to functions given by explicit formulas and teaches them how to evaluate functions. It provides them a foundation for further exploration. The geometrical discussion of lines is intended to develop their skill in using euclidean geometry and geometric intuition to determine the algebraic descriptions and properties of functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the meaning of restriction to subdomains and the basic operations with functions (sum, product, quotient, composition).

Identify the graph of a restricted function given the graph of the function. (Note: our functions are defined to be graphs and the language "graph of a function" just means "a graphical representation of the function".)

Application and Analysis (Analysis)

Calculate the iterated sums, products, quotients, and composites formed by multiple functions.

Break down a complicated function into simpler components in specified ways.

Evaluate and Create (Synthesis)

Determine the domain and range of some composite functions.

Summary.

Students learn how to construct compound functions and how to view complicated functions as being constructed from simpler functions. This is critical for them to later understand how transformations of the plane act on functions. It is also critical for them to understand later the various rules for differentiation.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Evaluate a piecewise defined function at given points.
Recognize what the partition is for any given piecewise defined function.
Describe the domain of a piecewise defined function.
Recall the meaning of a refinement of a partition.
Recall the meaning of commensurable partitions for two given piecewise defined functions.

Application and Analysis (Analysis)

Determine common refinements of two partitions.
Break down two given partitions for a piecewise defined function into a common refinement.
Use alternate presentations of two piecewise defined functions to compute their sum, product, and quotient.
Determine the range of a piecewise defined function (when reasonable).

Evaluate and Create (Synthesis)

Create functions with specified properties.
Determine the composite of two piecewise defined functions.
Generate the solution sets for equalities and inequalities involving piecewise defined functions.

Summary.

Students learn to work with piecewise defined functions by studying the properties of such functions on each of their defining regions. This section reinforces the students’ understanding of the domain and range of a function. It also improves their grasp of conjunction and disjunction. The exercises they encounter are complicated and working through such exercises will help them to better understand the use of decomposition as a problem solving tool: to decompose complicated problems into simpler ones that are more easily solved.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Evaluate real valued functions of several variables at specified points in their domains.

Match graphical representations of (very basic) real valued functions of several variables with the sketches of their graphs.

Application and Analysis (Analysis)

Determine the level sets of a function on a planar domain.

Sketch the graph of a function defined on the boundary of a square.

Give original examples of functions of two variables and sketches of their graphs.

Evaluate and Create (Synthesis)

Sketch the graph of a function dependent on the distance on the square from fixed point on the square.

Determine the formula for a function dependent on the distance on the square from fixed point on the square.

Summary.

Students find it difficult to understand the trigonometric functions as functions on a circle. They also have difficulties in understanding the difference between the domain and range of a function. By introducing functions of several variables or functions on domains embedded in the plane, we provide students with geometric examples that more clearly distinguish between the domain and the range of a function. We also have students early on in the course study the analogs of trigonometric functions on, for example, the square. This helps them to better understand the idea of a function on a circle. The main difficulty is that our examples at this point are very limited because they do not yet know, for example, what a circle is or why a plane should have the equation it has because our tools are limited. In this first lesson, we give only very simple examples. Once students know more about vectors and circles, polynomial graphs, rotations, and so on, we can come up with more interesting examples. As we develop tools, we will have more interesting examples that we could not otherwise discuss without this section.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize if a point is a solution to a system of linear equations.
Recognize if a point is a solution to a system of linear inequalities.

Application and Analysis (Analysis)

Solve any system of linear equations by either repeated substitution or by ellimination.
Determine when a system of linear equations has a solution and does not have a solution.
Graphically determine the feasible set of a single linear inequality in two variables.

Evaluate and Create (Synthesis)

Graphically determine the feasible set of multiple linear inequalities in two variables.
Formulate in a set theoretic language the feasible set of multiple linear inequalities in two variables using a graphical realization of the feasible set.

Summary.

While this section is mostly review, we focus on teaching students to use graphical representation of data to discover complicated algebraic relationships. The approach further reinforces students' understanding of logical conjunction. They will intersect multiple feasible sets to determine the feasible set of a system and determine the boundary of the feasible set. They improve their understanding of set builder notation by carefully writing the feasible set in set builder notation.

II. Transformation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize how an arrow in the plane acts on points in the plane.
Explain what it means for a vector to be a set of equivalent arrows.
Recall that the difference between points in the plane is a vector, that vector can sum and admit multiplication by a real number, but that points in the plane do not sum.
Identify a coordinate representation for a vector.

Application and Analysis (Analysis)

Use the coordinate representation of a vector to translate points and sets.

Evaluate and Create (Synthesis)

Determine the action of vectors on functions.
Determine the equation for the translation of a locus of points in the plane that satisfies a given equation.

Summary.

Translation is the first rigid motion of the plane that we study. We study translation by studying vectors and the action of vectors on the plane. This is our main tool for studying all other rigid motions of the plane and for studying the parameterization of lines.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize the geometric meaning of scaling a vector.
Recall how to scale a vector given in coordinates.
Recall the definition of distance in the plane.

Application and Analysis (Analysis)

Determine the equation of a circle given specifying information.
Calculate the polar form of a vector.
Determine graphically the asymmetric axial scalings of subsets of the plane acting on subsets of the plane.
Calculate the way in which an asymmetric scaling of the plane acts on a function.

Evaluate and Create (Synthesis)

Determine the equation for a symmetric and an asymmetric axial scaling of a locus of points in the plane that satisfies a given equation.
Determine the equation for a general asymmetric non-axial scaling of a locus of points in the plane that satisfies a given equation.

Summary.

Scalings of the plane are transformations that are very important in the study of functions. The allow us to more easily sketch the graphs of functions, they give us a vivid way of understanding tangency, they help us to determining the asymptotic properties of functions. They also give us a rigorous way of understanding the scaling of physical objects, which is critical to the following section.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the fundamental units of a physical quantity.
Recognize if two physical quantities are commensurable.
Identify the conversion factor between commensurable physical quantities.
Describe a physical quantity using different units,

Application and Analysis (Analysis)

Solve linear scaling problems.
Solve simple nonlinear scaling problems.

Evaluate and Create (Synthesis)

Solve general nonlinear scaling problems.
Solve problems involving Galileo’s square-cube law.

Summary.

Physical units arise up in any mathematical description of a physical experiment. We explore the both linear and nonlinear relationships in the scaling of physical quantities. We emphasize unit conversion and dimensional analysis. In settings where the relationships are not exact, we emphasize that the relationships we derive are not exact. For example, the change in the volume of paint on a sphere where the thickness of the paint remains the same but the radius changes is a complex relationship, however, the volume approximately changes as the surface area so long as the layer of paint is thin. The students should understand that we are investigating an approximation in these cases. Studying units will help students with applications and in deriving relationships between variables in application problems.

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Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the vector equation for a line.
Recall the form of a parameterization of a line.
Identify the vector that translates points along a given line given either two points on the line or the slope of the line.

Application and Analysis (Analysis)

Determine the position of a point on a line segment with information about absolute distance from another point on the line.
Determine the position of a point on a line segment with information about relative distance from other points on the line.
Determine from a transformational perspective the position of a particle that is at a point at one time and another point at another time.
Determine from a transformational perspective the equation of motion of a particle that moves on a given line in a certain direction at a certain speed.
Use vectors to find the midpoint of a line segment or a point that divides a line segment into segments with specified ratios of lengths.

Evaluate and Create (Synthesis)

Evaluate whether or not two particles in linear motion collide.
Formulate the equation of motion of a particle that moves in a piecewise linear fashion.
Determine the equation for a locus of points in the plane that is the translation of a locus of points satisfying a given equation.

Summary.

This section covers the basic ideas needed to understand and describe linear motion. This is a critical section in the flow of ideas in the course and provides students will the basic tools that they will need in many later sections, including the next. Students will need to be very fluent with their computational skills.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall the expression for a vector that is perpendicular to another vector.
Interpret the relative orientations of two orthogonal vectors.
Recall the slope of a line perpendicular to another line.
Recognize graphically the set of points that is the reflection across a line of a different collection of points.

<u>Application and Analysis</u> (Analysis)

Determine the equation of a line that is perpendicular to a given line.
Calculate the position of a point on a line that is closest to a point in the plane.
Create rectangles in the plane with specified side lengths or ratios of side lengths.
Formulate as a piecewise defined function in time the equation of motion of a particle moving on a rectangle in the plane.

<u>Evaluate and Create</u> (Synthesis)

Determine the reflection of a point in the plane across a given line.
Determine the equation for a locus of points in the plane that is the reflection of a locus of points satisfying a given equation.

Summary.

The main point of this section is for students to understand the meaning of reflection and how to compute the reflection of a point across a line. A critical application is in the next section, where they study inverse functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall that the reflection of the point (a, b) across the line given by $y = x$ is the point (b, a).

Recognize graphically why the horizontal line test for a planar function is the vertical line test for its reflection across $y = x$.

Recall the condition for invertibility of a function in a general setting.

Explain the relationship between the general condition for invertibility and its graphical interpretation in the planar setting in terms of reflection.

Application and Analysis (Analysis)

Calculate the formula for the inverse of some simple functions.

Determine points on the inverse function that correspond to specified points on the function.

Determine the set theoretic inverse of a function given a function that has certain specified points or that is given by a simple formula.

Use restriction to make a non-invertible function invertible on a smaller domains.

Evaluate and Create (Synthesis)

Create schemes to estimate the values of the inverse function at specified points given a function that is easy to evaluate.

Calculate the inverse of a function whose domain has been restricted to guarantee the function's invertibility.

Summary.

We use reflection to define and determine the inverse of a planar function and extend these definitions to more general settings of non-planar functions. The geometric perspective will help us later in studying tangency to inverse functions.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize the graphical interpretation of adding points on a circle.
Recall the group addition law for points on the unit circle in cartesian coordinates.
Recall the formula for rotation of points around the origin by a given angle.

Application and Analysis (Analysis)

Calculate the rotation of a point about an arbitrary point.

Evaluate and Create (Synthesis)

Determine the equation for a locus of points in the plane that is the rotation about a given point of a locus of points satisfying a given equation.

Summary.

We use basic ideas about translation and perpendicularity to derive the group addition law on the unit circle. We use scaling and translation to extend the group addition law to define rotation about arbitrary points in the plane. Measurement of angles and the angle summation formulas of the trigonometric functions will follow immediately from our understanding of rotation.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall the definition of a fraction of a circle.
Recall the interpretation of an angle between rays as a point on the unit circle.
Recall the meaning of a measurement of an angle.
Match the sine, cosine, and tangent functions with their geometric realizations as y-coordinate, x-coordinate, and slope.
Recall the meaning of angle of elevation and angle of depression in surveying problems.

<u>Application and Analysis</u> (Analysis)

Determine the angle summation formulas from the group operation on the circle.
Determine the Pythagorean identities from the defining equation for the unit circle.
Solve for all points on the unit circle with a single specified coordinate.
Calculate parameterizations of paths given by the motion of a particle on a circle with specified constant speed.
Use graphical information to determine the angles associated to various points on a circle given information about the angles given by related points.
Use polar coordinates and convert from polar to cartesian coordinates.

<u>Evaluate and Create</u> (Synthesis)

Derive the half angle formulas.
Create a scheme for approximating the location in cartesian coordinates of points on a circle with certain angle measures.
Solve complicated surveying problems.

Summary.

Starting from an understanding of rotation and the group structure of a circle, we develop a basic understanding of the trigonometric functions. We will later explore inverse trigonometric functions, periodicity, waves, and so on.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Interpret graphically rotation by half of a circle.
Interpret graphically reflection across the axes.
Interpret graphically the inversion of the y-axis by the map $y \mapsto \frac{1}{y}$ for all (x, y) with $y \neq 0$.

Application and Analysis (Analysis)

Determine the action of the above three involutions on planar functions.

Evaluate and Create (Synthesis)

Sketch functions determined by the reciprocals of monomials using inversion of the y-axis.

Summary.

The current section puts us in a position to study rational functions by showing students how to use transformations to sketch reciprocals of monomials. It also sets the stage for studying symmetries of a function like evenness and oddness.

III. Rigidity

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall that lines are determined by two points.
Identify parameterizations of lines in three dimensions.
Identify equations for planes.
Recall the dot product of two vectors.

Application and Analysis (Analysis)

Determine the intersection of lines and planes.
Parameterize the intersection of planes in three dimensions.
Use the dot product to determine perpendicularity.

Evaluate and Create (Synthesis)

Determine planes by a normal vector and a point.

Summary.

Determination of lines and planes as well as determination of intersections of lines and planes give us our first examples of the utility of rigidity principles.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Match quadratic functions with their graphs.
Identify which quadratic polynomials have a maximum and which have a minimum.
Recall the quadratic formula.
Recall the factor theorem.
Identify the remainder of a polynomial on division by a linear polynomial.
Identify the zeros and the orders of zeros of a polynomial function in factored form.
Identify the asymptotic behavior of a polynomial.

Application and Analysis (Analysis)

Determine the maximum possible zeros and minimum possible zeros of a polynomial of a given degree.
Apply the factor theorem to long division.
Use transformations to derive the quadratic formula.

Evaluate and Create (Synthesis)

Sketch the graph of a polynomial function using local and global data.
Determine the solution to optimization problems involving quadratic functions: how close do boats in linear motion get to each other, what is the closest distance from a point to a line, maximize area bounded by a polygon of fixed perimeter, etc.

Summary.

This section covers polynomial functions from the point of view of transformation and rigidity. The factor theorem is really a rigidity result because of the uniqueness of the decomposition. Up to a scaling of the y-axis, a polynomial written in factored form is really determined by its zeros and the degrees of its zeros. This affords us a simple way of sketching polynomials in factored form. Later, we will develop a notion of tangency by studying the intersection of lines with polynomials. That viewpoint will be essential in later understanding tangency in the transcendental setting.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the zeros and polls (and their orders) of a rational function whose numerator and denominator are written in factored form.
Identify the asymptotic behavior of a rational function.

Application and Analysis (Analysis)

Present a rational function as a polynomial plus a rational function in proper form.
Sketch the reciprocal of a factored polynomial using inversion of the y-axis.

Evaluate and Create (Synthesis)

Use the local and global properties of a rational function given by a quotient of two polynomials in factored form to sketch the function.
Use the sketch of a rational function to determine its properties.

Summary.

This section studies rational functions by using the rigidity of such functions and their local and global behavior to graphically analyze them.

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Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify commensurable partitions for piecewise rational functions.

Application and Analysis (Analysis)

Use the graphs of polynomial and rational functions to determine the solutions to inequalities.
Calculate the sum, product, and quotient of piecewise rational functions.
Use the graphs of polynomial and rational functions to determine the solutions to compound inequalities.

Evaluate and Create (Synthesis)

Compose piecewise rational functions.
Solve inequalities involving piecewise rational functions using graphical methods.

Summary.

This section emphasizes the power of spatial reasoning in solving complicated inequalities involving piecewise rational functions. It builds students’ ability to use spatial reasoning to solve computationally difficult problems.

IV. Symmetry

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the symmetries of various sets.
Recall the definition of an odd and even function.
Identify even and odd functions from their graphical representations.
Identify the transformations under which even and odd functions are symmetric.

Application and Analysis (Analysis)

Determine the symmetry groups associated to different sets.
Demonstrate computationally whether a function is even or odd.

Evaluate and Create (Synthesis)

Argue that a given function is neither even nor odd.

Summary.

This section explores the symmetry of functions under reflection across the y-axis (even functions) and under rotation by half a circle (odd functions). It gives students an introduction to symmetry and the idea of a symmetry group.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a period of a functions and a periodic function.
Recognize periodicity and periods from the sketch of a function.
Match the trigonometric functions with graphical representations of subsets of the plane.
Identify the domains where trigonometric functions are invertible.
Identify the intersections of a horizontal line with a periodic function given certain principle intersections.

Application and Analysis (Analysis)

Determine the graphical representation of reciprocal trigonometric functions using y-axis inversion.
Determine the graphical representation of sinusoidal functions.
Calculate composites of trigonometric and inverse trigonometric functions.

Evaluate and Create (Synthesis)

Solve equations involving trigonometric functions.
Generate simulations of traveling waves.
Present a superposition of two sinusoidal functions as a varying amplitude multiplying a sinusoidal function.

Summary.

This section goes more in depth into the solution of trigonometric equations and the graphical representation of trigonometric functions. It also studies the application of trigonometric functions to the description of waves.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Identify whether a function satisfies a linear or exponential model of change.
Recall the various properties of exponential functions and logarithms.
Recall the definition of half-life, growth rate, and decay rate.
Identify the graphical representations of exponentials and logarithms.

<u>Application and Analysis</u> (Analysis)

Calculate simple logarithms.
Solve simple exponential and logarithmic equations.

<u>Evaluate and Create</u> (Synthesis)

Formulate models of change given data that follows an exponential or linear model of change.
Calculate half-line (or any "fractional life" or doubling life, etc.) of a function satisfying an exponential growth model.
Determine future values of a function satisfying linear or exponential growth models given two values at different times. This is an application of a rigidity principle.

Summary.

We study exponential functions and logarithms from the point of view that exponential growth satisfies a certain symmetry in growth, they change by the same factor on time intervals of the same length. This symmetry condition imposes a rigidity condition on the class of such functions, namely, they are determined by their values at two different time points. Use of transformational principles obviates most calculations and dramatically simplifies solving the majority of applied problems.

Learning Goals.

Students will be able to:

<u>Remember and Understand</u> (Recall)

Recall the definition of tangency for polynomials.
Recall the definition of tangency for rational functions
Recall the power rule for taking derivatives from an algebraic perspective.

<u>Application and Analysis</u> (Analysis)

Determine the equations of lines that are tangent to polynomial curves at given points.
Determine the slope of lines that are tangent by employing basic rules.
Determine the equations of lines that are tangent to rational curves at given points.

<u>Evaluate and Create</u> (Synthesis)

Determine the reflections of lines off of planar curves.
Find all points on a curve where the tangency is greater than second degree.

Summary.

In the setting of polynomial functions, tangency is purely algebraic. Namely, a line L is tangent to a polynomial function f at $(a, f(a))$ if $f - L$ is a polynomial with a degree two or greater intersection at a. This makes tangency in the algebraic setting a very geometrically vivid concept. Tangency is really about a certain kind of asymmetry of the intersection under increasingly large symmetric scalings of the plane. A line L is tangent to a rational function f at $(a, f(a))$ if $f - L$ is a rational function whose numerator has a degree two or greater zero at a.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of tangency for roots.
Interpret the idea of tangency to an inverse function using symmetry under reflection.
Recall that tangential intersections are degree two intersections except for at most finitely many points.

Application and Analysis (Analysis)

Determine the equations of lines that are tangent to certain inverse functions, powers and rational functions, at given points.
Calculate points where a polynomial attains an extremal value.

Evaluate and Create (Synthesis)

Determine the reflections of light rays off mirrors that have the shape of rational functions and roots.

Summary.

We use a reflection symmetry of tangency to determine the lines tangent to roots. This allows us to determine the lines tangent to functions even when functions may have cusps and are not differentiable in the analytical sense. We explore this notion of tangency and see that there are only finitely many points on f where tangency is greater than a degree two tangency. In the next course, we will show how this notion of tangency can be extended to transcendental functions.

The Principles of Calculus I

Lecture Video and Worksheet 1

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

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Lecture Review

Lecture 25

Lecture 26

Lecture 27

Lecture 28

The Principles of Calculus I
Sample Final 1

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

The line L passes through the points $(2,3)$ and $(5,9)$.

(a) Find an equation for L.

(b) If $L_{\perp}$ is perpendicular to L, then what is the slope of $L_{\perp}$?

Problem C2.

Recall that 3 feet is one yard and 60 seconds is one minute. Calculate the acceleration $7\,\dfrac{\mathrm{ft}}{\mathrm{s}^2}$ in units of yards and minutes.

Problem C3.

Suppose that a line segment L passes through the points $(1,3)$ and $(2,5)$.

(a) Find the point on L that is a distance of 2 from the point $(1,3)$ and that lies to the right of $(1,3)$.

(b) Find the point on the line segment L whose distance from $(1,3)$ is one third the length of the line segment L.

Problem C4.

Suppose that L is a line of slope 5 that intersects the origin. Where does L intersect the unit circle?

Problem C5.

Take f to be the function given by

$$f(x) = |x+1|.$$

Write the function f explicitly as a piecewise defined function.

Problem C6.

Rotate the point $(2,3)$ by the angle $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ around the point $(1,5)$.

Problem C7.

Take f to be the function given by

$$f(x) = x^2 - 4x + 7.$$

What is the minimum y value for a point in f?

Problem C8.

Take f to be the function given by

$$f(x) = (x+2)^2(x-1)(x-4)^3.$$

Sketch f and then find all x with $f(x) \geq 0$.

Problem C9.

What is the leading term of the polynomial f given by

$$f(x) = (-2x+1)^3(3x+1)^2(x-8)^4(x-1)^{100}?$$

For some constant C and some natural number n, your answer should look like Cx^n.

Problem C10.

What fraction of a circle is an arc of $\frac{5}{3}$ radians? A Zuma-radian is a measure of an angle. There are 120 Zuma-radians in a circle. How many Zuma-radians is $\frac{5}{3}$ radians?

Problem C11.

Suppose that A and B are angles in the first quadrant and $A > B$. Put the correct symbol, $<$ or $>$, in the boxes below.

$$\cos(A) \ \boxed{} \ \cos(B)$$

$$\sin(A) \ \boxed{} \ \sin(B)$$

$$\tan(A) \ \boxed{} \ \tan(B)$$

Problem C12.

Suppose that $\log_2(A) = 3$, $\log_2(B) = 2$, and $\log_2(C) = 4$. Calculate $\log_2\left(\frac{AC^2}{B^5}\right)$.

B Level Questions

Problem B1.

Compute $(f+g)(x)$, where f and g are given by

$$f(x) = \begin{cases} 2x+5 & \text{if } x \leq -1 \\ 3x & \text{if } x > -1 \end{cases} \quad \text{and} \quad g(x) = |x+3|.$$

Problem B2.

There is a building in front of you. The angle of elevation from your position to the top of the building is $10°$. You walk 50 feet towards the building and measure the angle of elevation to now be $25°$. How tall is the building?

Problem B3.

(a) It takes 2 workers 5 hours to shovel 100 cubic feet of sand. How many hours does it take 7 workers to shovel 300 cubic feet of sand?

(b) A pyramid has a height of 5 feet and a surface area of 50 square feet. A larger pyramid with the same proportions has a height of 12 feet. What is its surface area?

(c) A pyramid has a height of 6 feet and a volume of 10 cubic feet. If a smaller pyramid of the same proportions has a volume of 2 cubic feet, what is its height?

A Level Questions

Problem A1.

At time zero, Boat A is initially at $(0,0)$ and Boat B is initially at $(1,4)$. Boat A has velocity vector $\langle 1,3\rangle$ and Boat B has velocity vector $\langle -1,2\rangle$. When are the boats closest to eachother and how far apart are they at this time? Be sure to include the proper units in your answer.

Problem A2.

Let L_1 be the line given by $y = 2x$ and let L_2 be the line given by $y = 3x + 1$. Find an equation of the line given by the reflection of L_2 across L_1. To clarify: L_1 is the "mirror" and L_2 is being reflected.

Problem A3.

Find all solutions to the inequality $|x^2 - 4| > 2$.

Principles of Calculus I
Sample Final 2

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

Suppose that L_1 is the line given by

$$y = -x + 7$$

and L_2 is the line given by

$$y = 4x - 3.$$

Where do L_1 and L_2 intersect?

Problem C2.

Sketch on a real number line the set $(-7, 9] \cap \Big([-10, 2] \cup (5, 10)\Big)$.

Problem C3.

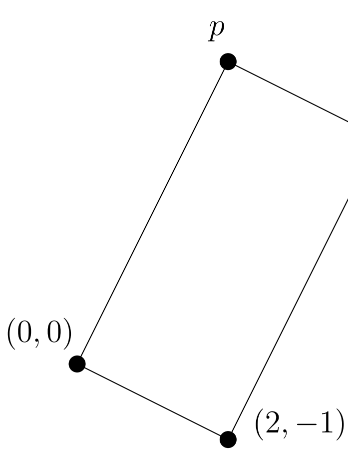
Let L be the line segment with endpoints $(1, 5)$ and $(6, 15)$. The point p lies on L. The distance from $(1, 5)$ to p is one fifth the distance of p to $(6, 15)$. What are the coordinates of p?

Problem C4.

The vector $\langle 2, 3\rangle$ translates points along the line L. The point $(1, 4)$ lies on L. Find the coordinates of the point on L that lies to the right of $(1, 4)$ and that is a distance of 7 from $(1, 4)$.

Problem C5.

The shape below is a rectangle, whose short side is half the length of the long side. Calculate the coordinates for p, q and write an equation for the line passing through p and q.



Problem C6.

Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } -2 < x \leq 1 \\ 3x - 2 & \text{if } x > 3. \end{cases}$$

What is the domain of f and the range of f?

Problem C7.

Suppose that f and g are given by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ 3x^2 & \text{if } 0 \leq x \leq 5 \\ 6x & \text{if } x > 5 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \cos(x) & \text{if } x < 2 \\ x^4 + 1 & \text{if } x \geq 2. \end{cases}$$

Write $f + g$ as a piecewise defined function.

Problem C8.

Find all solutions to the equation

$$(2\sin(3\theta) - 1)(4\cos(5\theta) + 1)(\sin(\theta) + 5) = 0.$$

Problem C9.

(a) Suppose that that $A(0) = 7$ and $A(1) = 3$. If A changes exponentially with respect to time, find a formula for $A(t)$ at any time t.

(b) Suppose that that $A(2) = 7$ and $A(5) = 3$. If A changes exponentially with respect to time, find a formula for $A(t)$ at any time t.

Problem C10.

Take f to be the polynomial given by

$$f(x) = (x + 5)^7(x + 2)(x - 3)^4.$$

Sketch f and find all x with $f(x) \leq 0$.

Problem C11.

(a) Where does the line $y = 10x$ intersect the unit circle?

(b) $\cos(\tan^{-1}(10)) =$

(c) $\sin(\tan^{-1}(10)) =$

Problem C12.

Calculate:

(a) $\arccos\left(\sin\left(-\frac{\pi}{7}\right)\right)$;

(b) $\arcsin\left(\sin\left(\frac{4\pi}{9}\right)\right)$.

B Level Questions

Problem B1.

Write the feasible set of the system

$$\begin{cases} x + y \leq 1 \\ y - 2x < 4 \\ y - x \geq 1 \end{cases}$$

in set builder notation.

Problem B2.

(a) Take f to be the rational function given by

$$f(x) = \frac{(x+5)^9(x+3)^4(x-2)^3}{(x+7)^3(x-1)^2(x-6)^4}.$$

Sketch f. Be sure to emphasize the global and local behavior of f and to NOT sketch f to scale.

(b) Let h be the function given by $h(x) = |f(x)|$. Write h as a piecewise defined function.

Problem B3.

A function A is changing exponentially with respect to time. It models the decay of a certain radioactive substance. If $A(0) = 12$ and $A(5) = 7$ then what is the half-life of the substance? Units of time are measured here in hours.

A Level Questions

Problem A1.

Let f be the function defined by the formula

$$f(x) = x^3 - 4x + 1.$$

What is an equation of the line tangent to the graph of f at the point $(2, 1)$? Be sure to directly use the algebraic definition of tangency—you will earn **no credit** for using other techniques.

Problem A2.

Suppose that f and g are the functions given by

$$f(x) = \begin{cases} 5x^2 & \text{if } x < 1 \\ 2x & \text{if } x > 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 5 - 2x & \text{if } x < 3 \\ x - 1 & \text{if } x \geq 6. \end{cases}$$

Write $f \circ g$ as a piecewise defined function.

Problem A3.

You swim to a buoy and then back. You can swim at a speed of 2 miles per hour with respect to still water. There is a current moving from the direction of the buoy to the shore at a speed s. When you are swimming against the current, you will naturally swim more slowly with respect to the stationary buoy, but when you swim back to shore, you will move more quickly with respect to the buoy. The buoy is a distance of 2 miles from the shore. For what value of s will your total swimming time be minimal? Does your answer depend on the buoy's distance or on your speed?

Course Outline – Common Features

Common Features

Lecture: three hours per week.

Discussion sections: two hours per week. One is a collaborative learning workshop. One is a directed question and answer period.

Homework: has two parts. The first part consists of self-check problems delivered electronically. The second part consists of careful write ups of the worksheets that have previously undergone guided peer review.

Exams: One midterm and one final.

Video: Video content supplements discussion section content and should be watched prior to discussion section.

Course Outline

The Principles of Calculus II

- V. Finite Approximation
- VI. Local Linear Approximation of Functions

Short Description:

The Principles of Calculus II applies the principle of finite approximation to the study of the local linear approximation of functions. Topics include: finite approximation of planar area, sequences and their limits, analysis of error; continuous limits; continuity; asymptotic behavior; approximating rate of change; the derivative; Newton’s Method; approximation by the tangent line; derivatives of elementary functions; implicit differentiation; related rates; the geometry of particle motion; the mean value theorem; extremal points; the antiderivative; simple first order differential equations.

Principles of Calculus II

• VI. Local Linear Approximation of Functions

The Principles of Calculus II

V. Finite Approximation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of area for a rectangle.
Recall the definition of a rectangular and triangular boundary.
Recall the definition of an oriented polygonal curve.

Application and Analysis (Analysis)

Calculate the area of a triangle using the shoelace formula.
Calculate the area of a triangle using Heron's formula.
Determine the orientation of a rectangular and triangular curve.

Evaluate and Create (Synthesis)

Calculate the circumcircle of a triangle.
Identify whether a point is inside or outside quadrilateral.

Summary.

Students develop a notion of the area bounded by a rectangle and a triangle. These are the “simple” shapes that will eventually give rise to a notion of area in the plane bounded by certain curves.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a polygonal Jordan curve.
Recall the statement of the Jordan Curve Theorem for polygonal curves.
Recall the meaning of the area of a triangle and the surveyor’s (or shoelace) formula.
Identify a triangulation of a polygonal Jordan curve.

Application and Analysis (Analysis)

Demonstrate the existence of triangulations of a polygonal Jordan Curve.
Calculate the area of a simple closed polygonal curve using the surveyor’s (or shoelace) formula.

Evaluate and Create (Synthesis)

Derive the additivity of area defined for polygonal Jordan curves.
Determine that this notion of area coincides with our physical intuition.
Defend the non-triviality of the area problem for general Jordan curves.

Summary.

Students develop a notion of the area bounded by a Jordan Curve. Although difficult to state in generality because of the need to define the notion of a homeomorphism, which is beyond what the students can understand at this point, we can without difficulty state all relevant results for polygonal curves. The general problem about calculating the area bounded by a Jordan curve can be stated roughly, alerting the students to the fact that a more careful statement is required for further study. Since we will only work with Jordan curves that are given by either graphs of functions together with three bounding line segments or, at the end of the course, for curves that are continuously differentiable except for finitely many points, our approach is sufficient. The key point is that students are already in the first lecture thinking about additive quantities, subdivision, superposition, and approximation of difficult to compute quantities by those that are more tractable, namely the area bounded by a Jordan curve as being approximated by a polygonal Jordan curve.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a sequence.
Recall the limit laws for sequences.
Recognize the limits for some simple sequences.
Explain non-rigorously why certain simple limits (ex. $\left(1+\frac{1}{n}\right) \to 1$) should be what they are and why certain limits (ex. $\left(1+\frac{1}{n}\right)^n \not\to 1$) are more difficult to compute.

Application and Analysis (Analysis)

Calculate simple limits using limit laws (including the squeeze theorem).
Calculate limits of recursively defined sequences.
Use error terms to compute more complicated limits.
Demonstrate when sequences have infinite limits.

Evaluate and Create (Synthesis)

Formulate statements equivalent to the Archimedean property of $\mathbb{R}$.
Formulate statements equivalent to the finite intersection property of $\mathbb{R}$.
Justify the existence of limits using basic facts about the real numbers.
Create sequences with specified limits.
Argue that certain sequences do not have limits.

Summary.

The notion of a sequential limit will be discussed in a rigorous way. All ensuing discussion of limits will be rigorously formulated in terms of sequential limits. Sequential limits are more concrete and easier to conceptualize. While this approach is very efficient, the efficiency comes at a cost. Namely, formulating the notion of a continuous limit requires a universal quantification over a collection of sequences. This is hard for students. However, limit laws become automatic in the continuous case from the sequential limit laws and showing the failure of the existence of a continuous limit is naturally framed as finding a specific sequence with certain properties.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Define a fraction of a circle.
Use the nested intersection property to define irrational "fractions".
Define approximation procedures for determining the area bounded by a circle and its circumference.

Application and Analysis (Analysis)

Calculate limits for the trigonometric functions with respect to different angle measures.

Evaluate and Create (Synthesis)

Determine upper and lower bounds arising from approximations of a circle.

Summary.

We review the basic ideas of Archimedes' famous work and show how the area bounded by a circle is defined and how the length of an arc of a circle is defined.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the limits associated to some simple functions.
Recall that limiting values of a function do not depend on the function value at the limit point.
Recall the definition of a continuous limit in terms of sequential limits.
Recall the definition of a one sided limit in terms of sequential limits and restriction of domain.
Recall the definition of an infinite limit and a limit at infinity in terms of sequential limits.
Recognize that the limit laws of continuous limits come from the limit laws for sequences.

Application and Analysis (Analysis)

Use the error estimates to calculate limits.
Use limit laws to calculate limits.

Evaluate and Create (Synthesis)

Determine complicated limits.
Use sequences to find counterexamples to the existence of a limit or to the validity of a proposed limit.
Justify certain limits without using the $\delta - \varepsilon$ formalism and instead using the Landau notation and analysis of error terms.

Summary.

In the process of solving exercises in this section, students will already make the necessary calculations for determining the continuity of elementary functions and determining the derivatives of most of the basic elementary functions. This disentangles some of the technical difficulties from the later conceptual difficulties that students will face. For example, in the next section, they will need to show that sin, $\sqrt{\cdot}$, and other elementary function are continuous at every point x in their domains. In the previous section, they will have already estimated $\sin(x+h) - \sin(x)$, $\sqrt{x+h} - \sqrt{x}$, and other such differences. In this section, they will use their estimates from previous sections to calculate limits associated to the differences. In the next section, they will use limits to make and prove statements about continuity.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

- Recall the definition of continuity.
- Recall the intermediate value theorem and the fact that continuous functions attain their maximum and minimum values on closed and bounded intervals.
- Recognize a basic library of continuous functions.
- Identify the basic algebraic properties of continuous functions as properties that arise from the limit laws.

Application and Analysis (Analysis)

- Determine the asymptotic behavior of rational functions.
- Determine whether or not given functions (including piecewise defined functions) are continuous at specified points or on specified intervals.
- Demonstrate that certain functions are not continuous at specified points or on specified intervals.

Evaluate and Create (Synthesis)

- Generate functions with specified continuity properties and specified asymptotic properties.
- Create approximation schemes using the bisection method to find the solutions to equations.
- Justify the existence of solutions to equations involving continuous functions.

Summary.

Students learn to view continuity as a property that determines approximability. They learn to use limits to determine whether or not a function output is approximable at some point by nearby input values. They also learn to view this property graphically and use the properties of continuous functions to estimate solutions to equations involving continuous functions. Understanding the properties of continuous functions will be critical henceforth because we will generally restrict our study of functions to those that are continuous on open intervals.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recognize proper usage of the Landau symbols.
Describe error terms coming from differences of the form $f(x + h) - f(x)$.
Recall that differences of the form $f(x + h) - f(x)$ are differences coming from perturbation of input values.

Application and Analysis (Analysis)

Calculate upper and lower bounds arising from absolute values of functions.
Analyze error terms coming from sums and products of functions.

Evaluate and Create (Synthesis)

Estimate error terms coming from perturbing quotients and composites of functions.
Illustrate graphically the errors due to perturbing inputs of a function.

Summary.

The only limits that we directly study will be limits that tend to zero and infinite limits. This section studies limits that tend to zero and so provides a tool for studying continuity and for studying differentiability. We purposely avoid the $\delta - \varepsilon$ formalism, but make sure that all technical tools are in place and conceptual tools are established for students to later make use of this formalism. The reason for this is that understanding the formalism is really a two step process. First, students must be able to perform the algebraic manipulations. Second, they must understand the more abstract logical basis for the formalism. The second is difficult for students who lack the basic technical skills. Focusing on the technical skills is sufficient for applications to calculus, whereas the formalism is required for rigorous proofs, which come in later classes.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of the difference quotient.
Recognize the relationship between the difference quotient and the average rate of change of a function on an interval.
Recognize the form of the difference quotient for polynomial functions.

Application and Analysis (Analysis)

Calculate the limit of the difference quotient for polynomial functions.
Calculate the limit of the difference quotient for rational functions.

Evaluate and Create (Synthesis)

Determine using limits the slopes of lines tangent to rational functions at specified points on the function.
Determine the linear approximation a given rational function using the definition of the derivative as a slope.

Summary.

The notion of the derivative for general functions extends the notion of the derivative for polynomials and rational functions. To understand this more general notion and have a method for calculating derivatives of general functions, we first relate the difference quotient and its limit to the local linear approximation of polynomial and rational functions. In the next section, we will develop the idea of the local linear approximation for more general functions.

• To Outline
• To Lectures

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a sequence of partial sums.
Recognize the convergence of some common series.

Application and Analysis (Analysis)

Use the comparison test, alternating series test, and limit comparison test.

Evaluate and Create (Synthesis)

Analyze the exponential function as an infinite sum.

Summary.

This section introduces the notion of an infinite series and studies the convergence of such series. It also presents the exponential function as a differentiable function.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of rectifiable.
Recall the definition of the Riemann integral.

Application and Analysis (Analysis)

Use the Riemann integral to calculate areas.
Use series to approximate the arclength of a curve.
Use series and the area bounded by polygonal Jordan curves to approximate the area bounded by a Jordan Curve.
Use series to define the Riemann Integral.

Evaluate and Create (Synthesis)

Determine the area bounded by a Jordan curve defined by the graph of a function for some specific functions.
Contrast the simplified situation of calculating the area bounded by a Jordan curve defined by the graph of a continuous function with the area bounded by a general Jordan curve.

Summary.

In this section we develop the notion of the Riemann Integral and show that continuous functions are Riemann integrable over closed intervals. The introduction of the integral at this early stage follows the historical development of the subject as well as Courant's approach in his calculus courses.

VI. Local Linear Approximation of Functions

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of the derivative of a function in terms of the local linear approximation of a function.

Recognize the connection between the limit of the difference quotient definition of the derivative and the definition in terms of local linear approximation.

Recall the basic differentiation rules for compound functions.

Application and Analysis (Analysis)

Calculate the derivatives of elementary functions.

Calculate the derivatives of compound functions using the basic rules of differentiation.

Approximate the value of a differentiable function near a known value.

Evaluate and Create (Synthesis)

Derive the basic differentiation rules using the definition of a derivative that comes from the local linear approximation of a function.

Approximate a root of a polynomial function that lies in a certain region using Newton's method.

Summary.

Rather than wait until the end of a course on differentiation to define the local linear approximation of a function as is usually done in calculus courses, we make this the primary tool for defining the derivative of a function. We pay careful attention to the error terms that are typically ignored in calculus courses. The basic rules and properties of differentiation are immediate consequences of the analysis of error terms that students already understand. Connecting the notion of a derivative of a general function to that of a polynomial function makes the geometric intuition underlying idea vivid. We introduce Newton's method at this point as well as the basic differentiation rules not only because it makes conceptual sense to do so, but because introducing these early on gives us a a greater variety of interesting examples to present as exercises for reinforcing learning of subsequent topics.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify the derivatives of the basic (non-compound) elementary functions.
Identify useful ways to decompose a compound function for the sake of calculating its derivative.
Recognize the locus of points satisfying certain equations.
Recall the statement of the implicit function theorem in one variable.

Application and Analysis (Analysis)

Calculate the derivatives of elementary functions including roots, exponentials, logarithms, and the circular trigonometric functions.
Calculate the derivative of the inverse of a differentiable function.
Calculate the derivatives of compound functions using the basic rules of differentiation.
Approximate the value of a roots and of differentiable transcendental functions near a known value.
Calculate the derivatives of implicitly defined functions.
Calculate the lines tangent to curves given by the locus of point that solve certain equations.
Use logarithmic differentiation to differentiate functions.

Evaluate and Create (Synthesis)

Justify the critical estimates necessary for calculating the derivatives of the trigonometric functions.
Derive the formula for the derivative of an inverse of a differentiable function.
Justify the validity of the inverse function theorem using reflections and the local linear approximation of a function.
Model covarying quantities and linearize the equations that relate them.

Summary.

We carefully study the necessary estimates for determining the derivatives of the elementary functions. Students will master the basic calculation tools for calculating the derivatives of elementary functions. As examples, they will use Newton's method to approximate roots of transcendental equations, calculate derivatives of inverse functions including the inverse trigonometric functions, and approximate the values of elementary functions near certain known values.The main theoretical tool in this section is the implicit function theorem. Students will learn how to use this theorem, its applications, and its limitations. This section is primarily a section on applications of differentiation to solve real world problems.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall that continuous functions attain their maximum and minimum values on closed and bounded intervals.
Recall Fermat's theorem, Rolles' Theorem, and the Mean Value Theorem.
Recall L'Hopital's Rule.
Identify limits that have indeterminate forms.

Application and Analysis (Analysis)

Calculate the antiderivatives of a function that is the derivative of a given function.
Calculate the extremal values of a differentiable function.
Calculate the minimum and maximum values of a differentiable function on a closed and bounded interval.
Use L'Hopital's rule calculate limits that have indeterminate forms.

Evaluate and Create (Synthesis)

Justify the fact that if the derivatives of two functions are equal on an interval, then the functions are equal up to a constant.
Develop mathematical models of physical systems to optimize measurable quantities.
Justify error estimates using the bounds on the derivatives of a function.

Summary.

This section explores the rigidity properties of differentiable functions. The mean value theorem establishes this rigidity. The mean value theorem will be critical to our proof of the fundamental theorem of calculus.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a second derivative.
Calculate second derivatives.
Determine if a function is twice differentiable at a point.

Application and Analysis (Analysis)

Analyze where a given function is increasing and decreasing from a sketch of the function.
Analyze from a sketch of a function where the functions slope is increasing.
Analyze inflection points of a function from a sketch of the function.

Evaluate and Create (Synthesis)

Approximate the change in slope using a second derivative.

Summary.

This section explores how information about the second derivative determines the shape of a function.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall what it means for a function to satisfy a (simple) first order differential equation.

Recognize a phase portrait for a differential equation.

Application and Analysis (Analysis)

Solve a simple differential equation using only the uniqueness of antiderivatives up to a constant.

Calculate the trajectories of ballistics.

Evaluate and Create (Synthesis)

Approximate the solution of a linear differential equation using successive local linear approximations.

Summary.

This section explores the connection between the local properties of a differentiable function and the global properties of the function. The mean value theorem establishes the fundamental relationship that is really a rigidity principle. The mean value theorem will be critical to our proof of the uniqueness of solutions of some simple differential equations.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition the derivative of a curve in the plane and in three dimensional space.

Recall that a state in a classical mechanical system is a position together with a velocity vector.

Describe parameterizations of standard surfaces and of any surface given by the graph of a function on the plane.

Application and Analysis (Analysis)

Calculate the velocity vectors associated to curves in the plane and in three dimensional space.

Calculate the velocity vectors associated to particles whose motion is restricted to a surface.

Determine the speed at which a particle moves as a function of time given its path of motion.

Evaluate and Create (Synthesis)

Generate examples of motions of particles restricted to surfaces.

Generate the equation for the plane to a surface at a point.

Build graphical representations for the tangent plane of a surface at a point.

Summary.

While usually taught only in multivariable calculus courses, the study of the kinematics of particle motion is purely a one dimensional problem. Studying this motion will help us to better understand the arc-length of curves as discussed later, gives us a host of interesting applications of calculus, and is necessary for our eventual solution of our motivating question of determining the area bounded by a Jordan curve. Of course, we will restrict our study to continuously differentiable Jordan curves in the final section of our course.

The Principles of Calculus II

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Lecture 12– 13

Lecture 14– 15

Lecture 16

Lecture 17

Lecture 18

Lecture 19

Lecture 20

Lecture 21

Lecture 22– 23

Lecture 24

Lecture 25

Lecture 26

Lecture 27

Lecture 28

The Principles of Calculus II

Sample Final 1

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

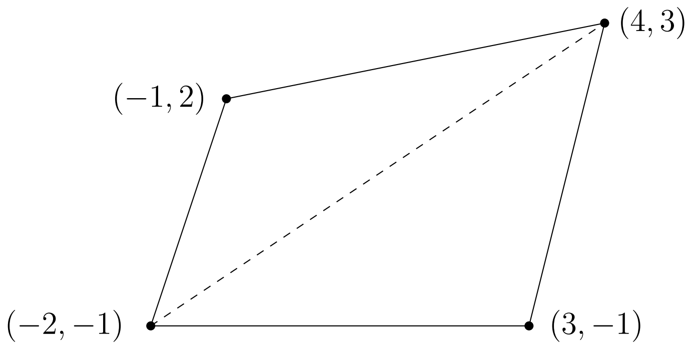
The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

Use the shoelace formula for the area of a triangle to determine the area of the given polygon.



The line segment $\overline{(-2,-1)(4,3)}$ divides the polygon above into two triangles. What is the altitude and area of each triangle?

Problem C2.

(a) Let $a_n = \frac{2n^3+n+2}{n^3-5}$. Calculate $\lim\limits_{n\to\infty} a_n$.

(b) Suppose that (a_n) is a sequence that is convergent to 0. Calculate $\lim_{n\to\infty} \frac{\sin(5a_n)}{3a_n}$.

Problem C3.

(a) Let f be the function defined by

$$f(x) = \begin{cases} 2x + a & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$$

Find a so that f is continuous.

(b) Let f be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x < a \\ 4x - 4 & \text{if } x \geq a. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\lim\limits_{x\to 0} \dfrac{\sqrt{x+9}-3}{x}$.

(b) Calculate $\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2+7}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = 2x - 1 + o(x^2) \quad \text{and} \quad g(x) = x + o(x).$$

(a) Calculate $(fg)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

(b) Calculate $(f \circ g)(x)$ and write your answer using the appropriate notation and the fewest possible number of symbols.

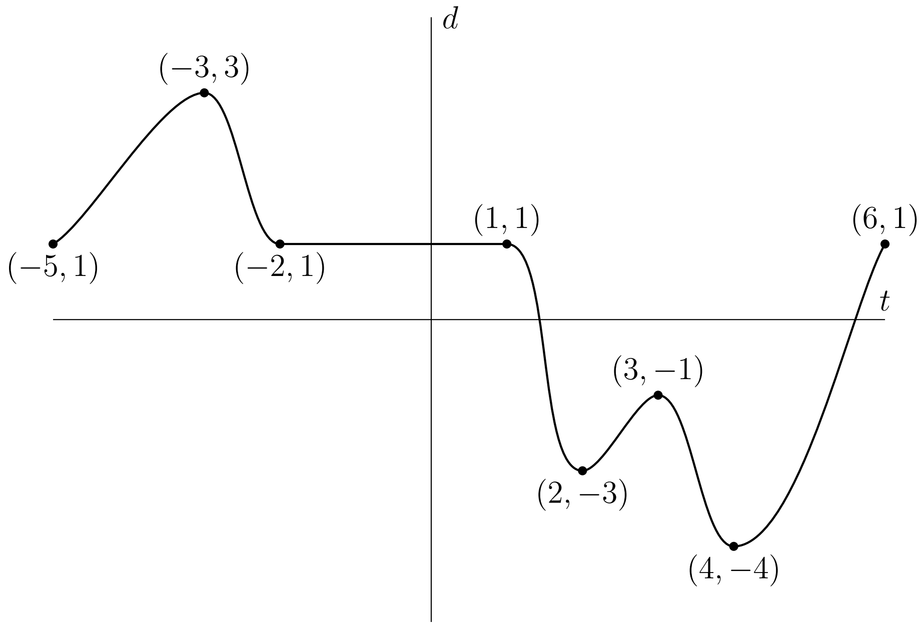
Problem C6.

(a) Calculate $\lim_{x\to 2^+} \frac{|x-2|}{x^2-4}$.

(b) Calculate $\lim_{x\to 2^-} \frac{|x-3|}{x^2-9}$.

Problem C7.

Below is a sketch of the position of a particle with respect to time. Measurements of the particle's position are taken on the time interval $[-5, 6]$.



(a) When is the particle at rest?

(b) When is particle moving in the positive direction?

(c) When is the particle moving in the negative direction?

Problem C8.

At time t, the position of a particle is given by γ, where

$$\gamma(t) = (t^2, t^3 + 1).$$

What is the velocity vector of the particle at time 5?

Problem C9.

Calculate the following limits:

(a) $\displaystyle \lim_{x \to 4^+} \frac{x^2 - 16}{\ln(x-4)}$;

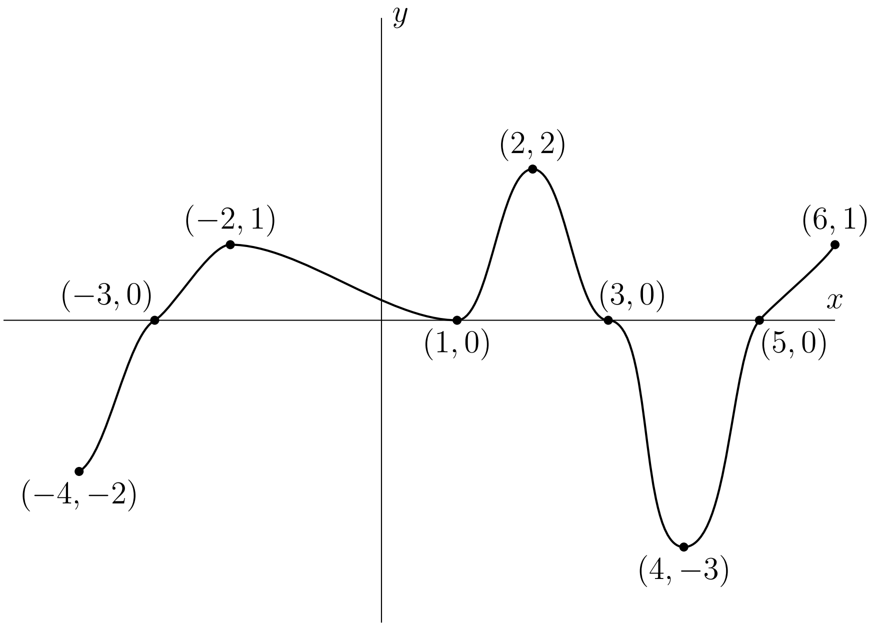
(b) $\displaystyle \lim_{x \to 1^+} (9x-9)^{\frac{1}{x-1}}$.

Problem C10.

Use the definition of the derivative to approximate the value of $\log_2(5)$.

Problem C11.

Suppose that f is a differentiable function and that this is a sketch of the graph of f':



List the extremal values of f in $(-4, 6)$ and determine if they are local maxima or local minima.

Problem C12.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying $z = 9 - x^2 - y^2$. A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (1+t, 2+t^2) \quad \text{with} \quad -\infty < t < \infty.$$

(a) What is the velocity vector of the particle at time $t = 0$?

(b) When $t = 0$, how fast is the particle moving?

B Level Questions

Problem B1.

Given the sequences (a_n) below, calculate $\lim_{n\to\infty} a_n$ and briefly justify your calculations:

(a) $a_n = 5^{\frac{1}{n}}$;

(b) $a_n = (5^n + 3^n)^{\frac{1}{n}}$.

Problem B2.

Three cubic inches per second of air is being pumped into a spherical balloon. How fast is the surface area changing when the radius is 12 inches?

Problem B3.

Show that the function f given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

is differentiable at 0. Calculate $f'(0)$.

A Level Questions

Problem A1.

The product fg is defined on all of $\mathbb{R}$ except possibly at finitely many points, differentiable where it is defined, defined at 0, and

$$f'(x)g(x) = x^2 - f(x)g'(x).$$

Given that

$$g(x) = 2x + 1 \quad \text{and} \quad f(0)g(0) = 1,$$

find a candidate for $f(x)$. Is your answer unique? Provide an explanation.

Problem A2.

A point particle is at the point $(1, 4)$ initially and travels with a constant velocity to the point $(3, 2)$, where it strikes the curve given by $y^3 = x^2 - 1$. It reflects off of the curve and continues onward indefinitely. The particle is always moving at unit speed. Describe the position of the particle as a function of time.

Problem A3.

A function f is differentiable on the interval $(1, 4)$ and $f(2) = 10$. The maximum value of $|f'|$ on $(1, 4)$ is 7. Find a positive real number E so that the values of f on $(2 - E, 2 + E)$ are guaranteed to lie in the interval $(8, 12)$.

The Principles of Calculus II

Sample Final 2

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

Let A be the region bounded by the curves $y = 0$, $x = 2$, $x = 10$, and $y = \ln(x)$. Use a Riemann sum approximation with an even partition with four intervals and a midpoint tagging to approximate the area of A.

Problem C2.

Suppose that $a_1 = 1$ and that $a_{n+1} = \sqrt{6 + a_n}$. Calculate $\lim\limits_{n\to\infty} a_n$. You may use the fact that the square root function is continuous, but be sure to justify that the limit exists.

Problem C3.

Let f be the function defined by

$$f(x) = \begin{cases} a & \text{if } x \leq 0 \\ \frac{\sin(2x)}{5x} & \text{if } x > 0. \end{cases}$$

Find a so that f is continuous.

Problem C4.

(a) Calculate $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x - 2}$.

(b) Calculate $\lim\limits_{x\to\infty} \dfrac{x^3 + x^2 - 1}{2 - 3x^3}$.

Problem C5.

For the problems below, use the standard Landau notation. Take f and g to be a function given by

$$f(x) = x + 1 + o(x - 2) \quad \text{and} \quad g(x) = 5x + o(x - 2).$$

Calculate $\lim\limits_{x\to 2} \dfrac{(f \circ g)(x) - (f \circ g)(2)}{x - 2}$ using the appropriate notation.

Problem C6.

Suppose that f is given by

$$f(x) = \begin{cases} \frac{x+2}{x+1} & \text{if } x \in (-\infty, -1) \cup (-1, 0) \\ 0 & \text{if } x = 0 \\ \frac{\sin(2x)}{x} & \text{if } x \in (0, \infty). \end{cases}$$

Calculate $\lim_{x \to 0} f(x)$.

Problem C7.

Suppose that $f(2) = 3$, $g(2) = 5$, $f'(2) = 3$, and $g'(2) = 4$. Suppose further that

$$h(x) = (f(x))^2 + f(x)\sqrt{g(x)}.$$

Calculate $h'(2)$.

Problem C8.

A particle moves counterclockwise along a circular track of radius 5. It makes one complete revolution every 3 seconds. What is its velocity vector at time $t = 1$?

Problem C9.

Calculate the following limits:

(a) $\lim_{x \to 0^+} x \log_2(x)$;

(b) $\lim_{x \to 0^+} x^{2x}$.

Problem C10.

A function f is differentiable on all of $\mathbb{R}$ and has the property that if x is a real number, then

$$f'(x) = 2x + 1.$$

If $f(0) = 2$, what is $f(x)$?

Problem C11.

The function f increases to the left of -2 and has a local maximum when $x = -2$. It decreases on $(-2, 1)$ and has a local minimum at $x = 1$. It increases on $(1, 3)$ and has another local maximum when $x = 3$. The function is decreasing on $(3, \infty)$. Sketch a function that can potentially correspond to f'.

Problem C12.

Let P be the plane given by the locus of points in $\mathbb{R}^3$ satisfying

$$z - 2x + 3y = 5.$$

A particle moves on P in such a way that the first two coordinates are given by

$$t \mapsto (t^2, t^3 + t - 1) \quad \text{with} \quad 0 \leq t \leq 4.$$

(a) What is the velocity vector of the particle at time $t = 1$?

(b) When $t = 1$, how fast is the particle moving upwards?

B Level Questions

Problem B1.

Calculate $\lim_{n\to\infty} \sqrt[3]{a+\frac{1}{n}}$. Justify your solution.

Problem B2.

Use Newton's method to estimate $\sqrt{7}$ by applying the method to a suitable quadratic polynomial. Stop at the third iteration of the method. Draw a picture that graphically illustrates what you are doing.

Problem B3.

For every time t, the velocity vector of a particle is given by

$$\gamma'(t) = \langle 1, t - 3t^2 \rangle.$$

At time 0, the particle is at the position $(0, 1)$. Where is the particle when $t = 2$? Use the appropriate theorems to justify your work.

A Level Questions

Problem A1.

Show that there is exactly one solution to the initial value problem

$$\begin{cases} y'(x) = 5y \\ y(0) = 1. \end{cases}$$

Hint: You will need to take the quotient of two solutions.

Problem A2.

Let P be the paraboloid given by the locus of points in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Find an equation of the plane that is tangent to P at $(1, 2, 4)$.

Problem A3.

Suppose that the function f is differentiable and that if x is a real number then $f'(x)$ is in $[-6, 2]$. Estimate $f(3 + h)$ if $f(3)$ is equal to 10.

Course Outline – Common Features

Common Features

Lecture: three hours per week.

Discussion sections: two hours per week. One is a collaborative learning workshop. One is a directed question and answer period.

Homework: has two parts. The first part consists of self-check problems delivered electronically. The second part consists of careful write ups of the worksheets that have previously undergone guided peer review.

Exams: One midterm and one final.

Video: Video content supplements discussion section content and should be watched prior to discussion section.

Course Outline

The Principles of Calculus III

- VII. Local Higher Order Approximation
- VIII. Integration

Short Description:

The Principles of Calculus III uses the higher order approximations of functions as well as the principles of decomposition and integration to study global properties of functions from their local behavior. Topics include: series and their convergence; Taylor's theorem and Taylor series; elementary differential equations of higher degree; the geometry of particle motion in several spatial variables; the fundamental theorems of calculus I and II and their consequences; integration techniques; area, volume, arc length, and surface area integrals in a single real variable; work integrals and planar area.

The Principles of Calculus III

VII. Local Higher Order Approximation

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a series of functions.
Recall the definition of a power series.
Recall the definition of radius of convergence.
Recognize some common convergent and divergent series of functions and power series.

Application and Analysis (Analysis)

Use the comparison test, limit comparison test, M test, and simplified versions of the root and ratio tests.
Use simplified versions of the root and ratio tests to determine the radius of convergence of a power series.
Use the alternating series test to determine the convergence of an alternating series and an alternating power series.

Evaluate and Create (Synthesis)

Estimate error terms for convergent alternating series.
Determine convergence using comparison tests.
Determine radius of convergence of a power series.

Summary.

This section presents the notion of convergence of a series as the convergence of the sequence of partial sums. This section is critical for the following section on Taylor series.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a Taylor polynomial and of Taylor's theorem.
Recall the meaning of higher order derivatives.
Recall the definition of the Taylor series for a function and the radius of con.

Application and Analysis (Analysis)

Calculate the Taylor series for a polynomial.
Calculate the Taylor series for a functions.
Use the second derivative test.
Determine the shape of a function based on concavity and extremal points (revisit the topic and emphasize non-algebraic functions as well as higher order information and osculating circles).
Determine the radius of convergence of a Taylor series.

Evaluate and Create (Synthesis)

Determine the error from remainder terms in Taylor's theorem.

Summary.

We begin our study of Taylor polynomials by studying the Taylor polynomial associated to a polynomial function, where there are only finitely many non-zero terms. We then extend this study to the study of Taylor polynomials and Taylor series for more general functions. This section is critical for understanding the higher order approximation of functions, for sketching functions, and for the approximations of integrals that will appear later.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the meaning of a higher order differential equation.
Recall some basic example of second order differential equations, for example, the equation $F = ma$.
Recall that two functions that are analytic on an open interval if they are equal on a convergent sequence in the open interval.

Application and Analysis (Analysis)

Determine the solutions to some simple higher order differential equations.

Evaluate and Create (Synthesis)

Create models of simple physical systems using higher order differential equations, for example, model an object falling with wind resistance.
Derive the main rigidity theorem for real analytic functions, that analytic functions that are zero on a convergent sequence are identically zero.

Summary.

This is mainly a section on applications that helps students to better understand the relevance of the topics they have studied. It prepares them for the more physically interesting examples of the next section.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of a higher order derivative of a curve.
Recall the meaning of force and acceleration for particles moving in the plane and in three dimensional Euclidean space.
Recall the definition of the dot product.
Calculate the component of force, velocity, and acceleration in a given direction.

Application and Analysis (Analysis)

Determine paths of motion of particles restricted to planes given by graphs of functions, including rotations around a point in the plane.
Determine paths of motion of particles restricted to surfaces given by graphs of functions.
Determine paths of motion of particles restricted to surfaces given by general surfaces.
Calculate the equation of the tangent plane to a surface at a given point.
Determine the tangent and normal bundles of a surface.

Evaluate and Create (Synthesis)

Calculate the component of acceleration of a particle moving on a surface that is tangent to the surface.
Calculate the normal acceleration of a particle whose motion is restricted to a surface.

Summary.

This is a culminating section on the geometry of particle motion. It reinforces the ideas needed for the final section of the course that deals with a simplified version of Green’s theorem and looks ahead to the multivariable calculus courses.

VIII. Integration

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the Fundamental Theorems of Calculus I and II.
Recall the Integral Mean Value Theorem
Recognize when integrals are Improper integrals.

Application and Analysis (Analysis)

Calculate the average value of a function on an interval.
Calculate the derivatives of integrals with variable bounds.
Use upper and lower Riemann sums, the midpoint rule, the trapezoid rule, and Simpson's rule to approximate the value of an integral.
Evaluate improper integrals.
Use the integral test for convergence of a series.

Evaluate and Create (Synthesis)

Determine estimates for the error terms arising from approximation of integrals.
Determine estimates for the error terms arising from improper integrals.

Summary.

We use the notion of the Riemann integral, already established in the first section of the course, to make rigorous the notion of the area bounded by a Jordan curve given by the graph of a Riemann integrable function. We prove the FTC I and II and use the integral to prove the existence of an antiderivative for any continuous function. Practice with differentiating integrals with bounds that are functions of a single real variable reinforce students understanding of both the chain rule and the fundamental theorem of calculus.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify when integration by parts is useful for calculating antiderivatives.
Recognize bijections between domains of integration.
Identify when substitution is useful for calculating antiderivatives.

Application and Analysis (Analysis)

Calculate basic examples of definite and indefinite integrals that use integration by parts.
Calculate basic examples of definite and indefinite integrals that use integration by parts.
Calculate basic examples of definite and indefinite integrals that use substitution.

Evaluate and Create (Synthesis)

Determine whether or not a real valued function on a subset of the line is a valid change of variables.
Justify the use of substitution in calculating definite and indefinite integrals.

Summary.

This section develops the main calculation tools in single variable integral calculus. We use integration by parts to prove Taylor's theorem with an integral remainder term. We carefully study the substitution theorem for Riemann integrals, correcting the usual incorrect presentation that views the integral as a directed integral and using instead the presentation that is in line with the theorem in the multivariable setting. We pay particular attention to identifying valid changes in variables. This approach is in line with the transformational approach of the precalculus course.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Identify polynomials that are irreducible in the real numbers.
Recall how to write a quadratic polynomial in scaled and shifted form.
Recall the change of variables needed to write the reciprocal of and quadratic polynomial in the form $\frac{1}{x^2+1}$.

Application and Analysis (Analysis)

Calculate the antiderivative of even powers of the cosine function.
Calculate the partial fraction decomposition of any rational fraction with a denominator in factored form.

Evaluate and Create (Synthesis)

Calculate indefinite and definite integrals of any rational function with a denominator in factored form.
Calculate improper integrals involving rational functions.

Summary.

Antiderivatives of elementary functions are seldom elementary. It is important that in teaching students integration techniques we make students aware of this fact. It is also important that they understand that calculating antiderivatives that are elementary functions is, for our intents and purposes, a solved problem. They should use and be encouraged to use symbolic solvers to do integration problems. They should also understand that large classes of problems can be easily solved by hand. Integrals of rational functions are among this class of problems. We teach the techniques with an emphasis on reinforcing students ability to apply the principle of decomposition, that is, the reduction of a single complicated problem into problems of a simpler type. Students should be aware that this is why we are teaching the topic at hand, not that it is independently important. Indeed, reduction of a problem to a partial fraction decomposition is a critical aspect of Risch's (semi) Algorithm that symbolic solvers employ.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definitions of the hyperbolic trigonometric functions and their relationship to the circular trigonometric functions.
Recognize integrals that can be solved by trigonometric and hyperbolic trigonometric substitution.
Recall the birational equivalence between the unit circle and the y-axis.
Recall the birational equivalence between the right half of the hyperbola given by $x^2 - y^2 = 1$ and the y-axis between -1 and 1.

Application and Analysis (Analysis)

Calculate basic examples of definite and indefinite integrals involving trigonometric integrals.
Calculate basic examples of definite and indefinite integrals involving hyperbolic trigonometric functions.

Evaluate and Create (Synthesis)

Justify the use of the Weierstrass substitution and the use of its hyperbolic analog.
Formulate more advanced trigonometric and hyperbolic trigonometric substitutions, where more careful attention needs to be paid to the domain and ranges of the functions, to calculate definite and indefinite integrals.
Develop mathematical models for ballistics with wind resistance.

Summary.

This section explores more technical applications of trigonometric and hyperbolic trigonometric functions in integration theory. Often presented as a a clever trick, the Weierstrass substitution is well motivated and comes from the birational equivalence of the circle and the y-axis. It identifies all rational points on the unit circle except $(-1, 0)$ with the rational points on the y-axis and gives a simple way of determining all Pythagorean triples. A similar transformation exists for the hyperbola given by the locus of points $x^2 - y^2 = 1$. These transformations are very useful in evaluating complicated integrals and more naturally motivates

the derivations of the anti-derivatives of the reciprocal trigonometric functions. We will use such substitutions to solve differential equations involving ballistics with wind resistance and compare the dynamics with the dynamics of ballistics using models without wind resistance.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the equation of the arc length element for a parameterized curve and the specialization to a curve given by a planar graph.
Match sketched regions of the plane to planar regions given by the intersection of feasible sets.
Identify symmetries of surfaces of revolution and the volumes they bound.
Recall the equation of the infinitesimal surface area element for a surface of revolution.
Recall the equation of the infinitesimal volume element for a surface of revolution.

Application and Analysis (Analysis)

Identify the intersection points of planar curves.
Calculate the area bounded by planar curves.
Calculate the arclength of a curve given by a planar graph.

Evaluate and Create (Synthesis)

Calculate the arclength of a parameterized curve in the plane and in three dimensional euclidean space.
Calculate the surface area of a surface of revolution.
Calculate the volume bounded by a surface of revolution.

Summary.

The topics of this section are rather standard, however, we pay particular attention to the identification of symmetries of surfaces of revolution to identify appropriate infinitesimal surface area and volume elements. Since we know that distance is infinitesimally equal to speed times time and we can calcualate the velocity vectors of both planar curves and curves in three dimensional Euclidean space, we can calculate arclength of any parameterized curve in these spaces.

Learning Goals.

Students will be able to:

Remember and Understand (Recall)

Recall the definition of work done by a force field.

Application and Analysis (Analysis)

Calculate the work done by a force field in moving a particle from one point in the plane or three dimensional Euclidean space to another.

Evaluate and Create (Synthesis)

Justify a simplified version of Green's theorem for certain vector fields.
Determine the area of a region determined by a Jordan curve with a continuously differentiable parameterization.

Summary.

We conclude the course with this section by calculating the area bounded by a continuously differentiable Jordan curve. The main point is that for vector fields $\frac{1}{2}\langle -y, x\rangle$, $\langle -y, 0\rangle$, or $\langle 0, x\rangle$ if Δ is a triangle and $\gamma\colon [0,1] \to \Delta$ is a counterclockwise oriented parameterization that is continuously differentiable at all but finitely many points, then the area of Δ is equal to $\int_0^1 F(\gamma) \cdot \gamma'(t)\,\mathrm{d}t$. The proof of Green's theorem for such vector fields requires only the tools that we have developed and solves an important and ancient problem.

The Principles of Calculus III

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6– 7

Lecture 8

Lecture 18

Lecture 19

Lecture 20

Lecture 21

Lecture 22

Lecture 23

Lecture 24

Lecture 25

Lecture 26

Lecture 27

Lecture 28

The Principles of Calculus III
Sample Final 1

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n}{2^n}$ convergent or divergent?

(b) Use the fact that

$$n^2 + 3n + 2 = (n+1)(n+2)$$

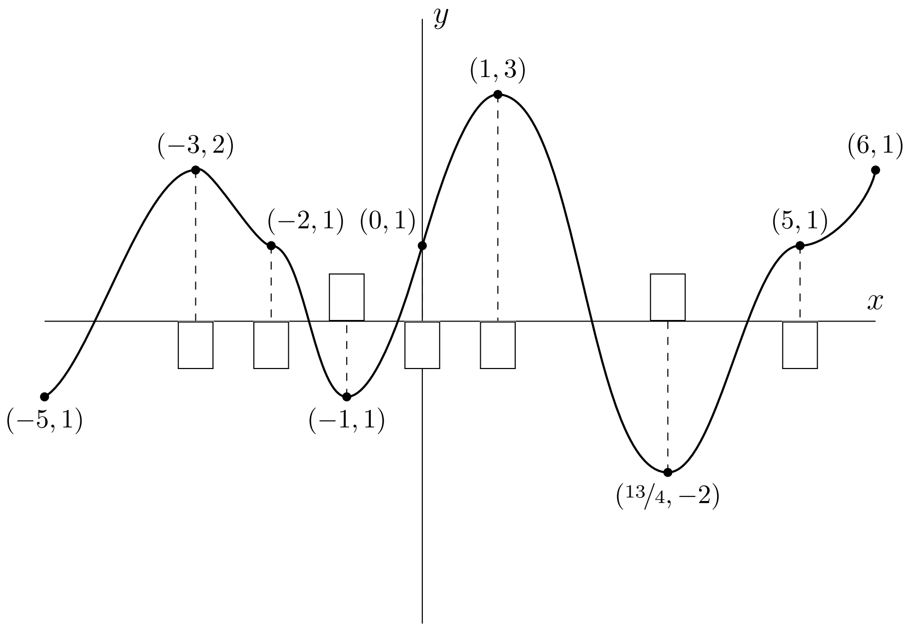
to calculate $\sum_{n=1}^{\infty} \frac{3}{n^2+3n+2}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n}}{n!}$?

Problem C3.

A sketch of the function f is given below.



Put the appropriate symbols $+$, $-$, or 0 in the given boxes to indicate the value of the second derivative at the x value of the given point in f.

Problem C4.

Calculate the Maclaurin series for f where

$$f(x) = 2x\sin(x^2).$$

Problem C5.

Solve the initial value problem

$$\begin{cases} f''(x) = 3x^2 - 1 \\ f'(1) = 2 \\ f(3) = 5. \end{cases}$$

Problem C6.

A particle moves along the segment of the line that passes through $(1, 2, 5)$ and $(3, 7, 12)$. At time $t = 0$, it is at $(1, 2, 5)$. The particle always moves to the right at a speed s, where for each non-negative real number t the function s is given by the equation

$$s(t) = 1 + t^2.$$

Find an equation for the position of the particle at time t.

Problem C7.

Calculate the average value of the function f on $[4, 9]$, where

$$f(x) = \frac{1}{x-2}.$$

Problem C8.

Determine whether or not the following integrals are convergent or divergent. Explain your reasoning.

(a) $\displaystyle\int_0^\infty \frac{1}{x^2 + 2x + 2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^\infty \frac{\sin(x)}{\sqrt{x^3}}\,\mathrm{d}x$

(c) $\displaystyle\int_0^\infty \frac{x}{x^2 + \mathrm{e}^{-x}}\,\mathrm{d}x$

(d) $\displaystyle\int_1^3 \frac{x\mathrm{e}^x}{x-1}\,\mathrm{d}x$

(e) $\displaystyle\int_1^4 \frac{x^2+2}{\sqrt{x-1}}\,\mathrm{d}x$

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\int_1^4 \sqrt{2+\sqrt{x}}\,\mathrm{d}x$;

(b) $\int_1^5 x\ln(x)\,\mathrm{d}x$;

Problem C10.

Interpret the following integrals as areas of regions, sketch the region, and use either basic geometry or transformations to calculate the integral.

(a)

$$\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x =$$

(b)

$$\int_0^1 x\,\mathrm{d}x =$$

(c) If f is continuous and invertible, $f(0)=0$, $f(1)=5$, and $\int_0^1 f(x)\,\mathrm{d}x = 2$, then

$$\int_0^5 f^{-1}(x)\,\mathrm{d}x =$$

Problem C11.

A particle moves in $\mathbb{R}^3$ and its position is given for each real number t by $c(t)$, where

$$c(t) = (t^2, \sin(t), 2t-5).$$

Set up but do not solve an integral that describes the length of the path that the particle traverses between time 0 and time 3.

Problem C12.

Use a transformation to turn the following integral into an integral of a rational function:

$$\int \frac{\sin(x)+\cos(x)}{1+\cos^2(x)}\,\mathrm{d}x.$$

Set up but do not evaluate integral that results from performing this substitution.

B Level Questions

Problem B1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 9 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (2t, t^2)$.

(a) Calculate the velocity vector of c at time $t = 1$ and its acceleration vector.

(b) What is the component of the acceleration vector in the direction of motion?

(c) What is the instantaneous rate of change of the speed of c at time t?

(d) What is the magnitude of the acceleration vector at time t?

Problem B2.

Solve the initial value problem

$$\begin{cases} y' = (2 - y)(1 + y) \\ y(0) = c. \end{cases}$$

Find a formula for y and then calculate $\lim_{x \to \infty} y(x)$ when $c = 1$ and when $c = 4$.

Problem B3.

The curves given by $y = (x - 1)^2$ and $y = x + 2$ bound a region R in the plane. Rotate R about the y-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let S be the surface given by $S = \{(x, y, x^2 + y^3) \colon (x, y) \in \mathbb{R}^2\}$. A particle of mass m moves along the surface so that the x–y coordinates are given for all t by $(2t - 1, t + 1)$.

(a) Find the equation of the plane that is tangent to S at the point $(1, 2, 9)$.

(b) What is the force on the particle in the direction of the upward pointing normal when the particle is at the point $(1, 2, 9)$?

(c) Set up (but do not evaluate) an integral that describes the length of the path that the particle traverses during the time interval $[1, 2]$.

(c) How much work does the surface do on the particle during the time interval $[1, 2]$?

Problem A2.

You control the motion of a point particle that moves in the plane. It has initial position $(0, 0)$ and initial velocity $\langle 10, -2\rangle$, where units of distance given in feet and units of time are given in seconds. The particle is equipped with accelerometers. The accelerometers give readings that can only be guaranteed to be accurate to $\pm\frac{1}{10}$ feet per square second. According to the accelerometers, the acceleration of the particle at time t is given as $a(t) = \langle 2t+1, t+\sin(t)\rangle$.

(a) According to the accelerometer data, where should the particle be at time t?

(b) Let $P(t)$ be the set of possible locations of the particle at time t. Given the error in the accelerometers, determine $P(t)$.

(c) What is the area of $P(t)$ as a function of time?

(d) The particle must not deviate by more than 10 feet from its predicted location or it will be impossible to accurately control. How often must precise location and velocity measurements be taken to ensure that accurate control is possible?

Problem A3.

Suppose that f has at least four continuous derivatives. Calculate $\int_0^1 f(x)\,\mathrm{d}x$ in the following way. Take an even partition of $[0, 1]$ with n steps. For each interval in the partition, integrate the first three terms of the Taylor expansion for f centered at the interval's midpoint.

(a) What is the area given by your approximation for each interval in the partition?

(b) Write down a formula for this approximation method. Be sure to simplify as much as possible so that you obtain more usable formula. Be sure to look for and eliminate any terms that are guaranteed to not contribute to the value of the integral.

(c) If the maximum value of the fourth derivative of f is M, then what is the maximum possible error in the approximation of the integral?

The Principles of Calculus III

Sample Final 2

Notes, books, calculators, and computing resources may be used in this examination, but you may not seek unauthorized assistance.

You may not receive full credit for a correct answer if insufficient work is shown.

Do not simplify your answers unless specifically requested or further than specifically requested.

The exam contains 18 questions with 156 possible points.

(108 points – C - Level)	Questions C1–C12 are each worth **nine points**.
(24 points – B - Level)	Questions B1–B3 are each worth **eight points**.
(24 points – A - Level)	Questions A1–A3 are each worth **eight points**.

C Level Questions

Problem C1.

(a) Is $\sum_{n=1}^{\infty} \frac{n-1}{3n^2-2}$ convergent or divergent?

(b) Calculate $\sum_{n=1}^{\infty} \frac{1+3^n}{4^{n-1}}$.

Problem C2.

What is the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$? Check for convergence at the endpoints.

Problem C3.

Suppose that g is given for each real number x by

$$g(x) = 2 + \frac{1}{3}x + \frac{1}{10}x^2 - \frac{3}{20}x^3 + O(x^4).$$

Find the first three terms of the Maclaurin series for f where

$$f(x) = g(x)e^{2x}.$$

Problem C4.

Suppose that f is at least four times differentiable and that

$$f(2) = 3,\ f'(2) = 1,\ \ f''(2) = 5,\ \text{and}\ f'''(2) = 7.$$

Calculate the first four terms of the Taylor series for f centered at $x = 2$.

Problem C5.

Suppose that f and g are analytic functions on $\mathbb{R}$ and that for each natural number n,

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2} + g\left(\frac{1}{n}\right).$$

Given that $g(3) = 5$, calculate $f(3)$.

Problem C6.

A particle moves in the plane with velocity vector v given for each non-negative real t by

$$v(t) = \langle 2t + 1, 3t^2 \rangle.$$

When t is 3, the position of the particle is $(1, 5)$. What is the initial position of the particle (position at time $t = 0$)?

Problem C7.

Estimate $\int_2^{10} \ln(x)\,\mathrm{d}x$ using the midpoint rule with four intervals. Be sure to include in your estimate a bound on the error.

Problem C8.

Calculate $\frac{\mathrm{d}}{\mathrm{d}x} \int_{\sin(x)}^{x^2+1} \mathrm{e}^{s^2}\,\mathrm{d}s.$

Problem C9.

Calculate the following integrals but do not simplify your answers:

(a) $\int_0^1 x \cos(4 - x^2)\,\mathrm{d}x;$

(b) $\int_0^{\frac{\pi}{12}} x \sin(3x)\,\mathrm{d}x.$

Problem C10.

Use partial fractions to write the integral $\int \frac{1}{(x-2)(2x^2+3)(x^2+1)^2}\,\mathrm{d}x$ in a simplified form. Do not solve for the undetermined coefficients.

Problem C11.

Let f be the function given for each real number x by $f(x) = 4\sin(x)$. Calculate the length of the arc determined by the segment of f between $(0, 0)$ and $(\pi, 0)$.

Problem C12.

Use Green's theorem to calculate the area determined by $\int_0^1 x^2\,\mathrm{d}x$. Verify that your answer is correct.

B Level Questions

Problem B1.

A particle moves along the skin of the cone C with vertex at $(0, 0, 0)$ and whose projection onto the plane $x = 0$ is the set of points given by $z = |y|$. The particle makes a complete counter clockwise (when viewed form above) rotation around the cone one time each second and the vertical component of the particle's velocity vector is $\langle 0, 0, 2t\rangle$.

(a) What is the particle's velocity vector at time t?

(b) What is the particle's acceleration vector at time t?

(c) Set up but do not evaluate an integral that describes the distance the particle has traveled between time $t = 0$ and time $t = 2$.

Problem B2.

The curves given by $y = x^2$ and $y = x^3$ form a simple closed curve that bounds a region R in the plane. Set up but do not evaluate any integral used in this problem.

(a) What is the length of the curve?

(b) Determine the area bounded by R without using Green's theorem.

(c) Use Green's theorem to determine the area bounded by R.

Problem B3.

The curves given by $y = x^2 + 1$ and $y = x + 3$ bound a region R in the plane. Rotate R about the x-axis to form the solid D. Set up but do not evaluate any integral used in this problem.

(a) What is the volume of D?

(b) What is the surface area of D?

A Level Questions

Problem A1.

Let P be the surface given by the set of all triples (x, y, z) in $\mathbb{R}^3$ satisfying

$$z = 4 - x^2 - y^2.$$

Let c be the curve on P where $c(t) = (x(t), y(t), z(t))$ and where $(x(t), y(t)) = (t, 2t + 1)$. Imagine that P is a very thin, stationary, and nondeformable sheet and that a particle whose motion is determined by c is moving along and under P.

(a) When does the particle experience the greatest normal force?

(b) Where is the particle at this time?

Problem A2.

Let f be the function given by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 4. \end{cases}$$

Find a sufficiently small mesh size for an even partition for $[0, 4]$ so that you can approximate $\int_0^4 f(x)\,\mathrm{d}x$ to within $\frac{1}{10}$ of the actual value of the integral.

(a) What is a reasonable mesh size?

(b) What is the approximate value that you have calculated?

(c) What is the actual value of the Riemann Integral of the given function on $[0, 4]$.?

Problem A3.

Suppose that S is the surface given by

$$S = \{(x, y, z) \colon (x, y) \in \mathbb{R}^2,\ z = |y|\}.$$

If c is a differentiable curve on S given by $c(t) = (x(t), y(t), z(t))$, then what are the possible velocity vectors for c when $y = 0$?